Teachers' Knowledge of Students' Mathematical Learning:

An Examination of a Commonly Held Assumption

Dina Tirosh
Tel Aviv University

Ruhama Even
Weizmann Institute of Science

1

A common belief:

Teachers should be aware of and knowledgeable about students' mathematical learning.

This knowledge significantly contribute to many aspects of the practice of teaching.

2

Main issues:

1. The term: students' mathematical learning;

2. The validity of the assumption: different theoretical perspectives;

3. Teacher education programs that focus on different aspects of students' mathematical learning;

4. Issues for further research.

3

STUDENTS' MATHEMATICAL LEARNING

The Term:

1. **Student conceptions**;
2. Different forms of knowledge and kinds of understanding;
3. Classroom culture

4

Student conceptions

Theory building:

- (1) developing a theory that describes how students learn specific mathematical domains or concepts (e.g., the van Hiele theory);
- (2) constructing theories that suggest general principles for learning mathematics (e.g., the acquisition of mathematical concepts – Davis, 1975; Dubinsky, 1991; Gray & Tall, 1994; Sfard, 1991).

5

Student conceptions

Misconceptions and errors:

- (1) Focus on specific "local" concepts: describing in detail errors that students make in specific topics (e.g., Gutiérrez & Boero, 2006; Nesher & Kilpatrick, 1990).
- (2) The evolution of misconceptions with age and instruction (e.g., Fischbein & Schnarch,1997; Hershkowitz,1987; Vosniadou & Verschaffel, 2004).

6

Student conceptions

General, underlying sources of students' incorrect responses:

- The Conceptual Change Approach
- Concept Image, Concept Definition
- The Intuitive Rules Theory (Stavy & Tirosh, 2000).

7

Students' conceptions

Exploring what students know and can do:

A newer trend. Highlighting the continuity in knowledge between novices and masters.

Smith, diSessa, and Roschelle (1993) showed fundamental similarities in characteristics of masters' and novices' knowledge about fractions: Both groups constructed strategies tailored to solving specific classes of problems instead of using the more general school-taught strategies.

8

STUDENTS' MATHEMATICAL LEARNING

The Term:

1. Student conceptions;
2. **Different forms of knowledge and kinds of understanding;**
3. Classroom culture

9

Different Forms of Knowledge and Kinds of Understanding

- Instrumental, relational,
- Conceptual, procedural,
- Implicit, explicit,
- Elementary, advanced,
- Algorithmic, formal, intuitive,
- Visual, situated,
- Knowing that, knowing how, knowing why, knowing to

10

Classroom Culture

A shift from examining human mental functioning in isolation

to

Considering cultural, institutional and historical factors – the nature of the class (socio-mathematical norms).

11

Classroom Culture

Aim: A different mathematical culture in the classroom

(Roles and responsibilities of teacher and students in classroom discourse).

Research: Social and the sociomathematical norms in classroom cultures (Yackel & Cobb, 1996).

(Identity – Traditionally: Learners' self concept and self efficacy – Currently: Reference to socio-cultural notions (e.g., Lerman, 2006).

12

SHOULD TEACHERS

BE AWARE

OF STUDENTS'

MATHEMATICAL LEARNING?

13

Knowledge about Students' Conceptions?

Behaviorism

- A basic assumption: It is impossible for anyone (including teachers) to know what goes on in the students' mind;
- Teachers: Determine the correctness of the student's response, not the student's conceptions

14

Knowledge about Students' Conceptions?

Constructivism

Children's knowledge is qualitatively different from that of the adult;

The teacher should attend to students' thinking, form an adequate model of students' ways of viewing an idea, and then construct a tentative path along which students may move to construct a mathematical idea;

The essence of constructivism is to know and understand student conceptions.

15

Knowledge about Students' Conceptions?

Situationist perspective:

Learning: A process that takes place in a participative framework, not in an isolated individual mind.

Knowing: The practices of a community and the abilities of individuals to participate in those practices;

The situationist perspective attends to students' ability to participate in shared mathematical activities.

16

Knowledge about Forms of Knowledge?

- Behaviorists: Knowledge - an organized accumulation of facts and procedures. These are the types of knowledge that teachers are apt to emphasize in instruction.
- Constructivist: The development of conceptual knowledge, problem solving strategies, and meta-cognitive abilities. Teachers should be knowledgeable about different forms of knowledge.
- Situationist: Attention to participation in activities, which involve the use of different forms of knowledge.

17

Knowing about Classroom Culture?

- **Behaviorism**: The teacher presents correct procedures and provides opportunities for practice. The classroom is a collection of individual students.
- **Constructivist**: Learning environments: provide students with opportunities to construct conceptual understanding and to foster problem-solving and reasoning abilities. Constructivism focuses on the individual student.
- **Situationist**: The classroom culture is the essence. Teachers exemplify valued practices, encourage the development of desired norms, and guide students as they become increasingly competent practitioners.

18

Some Issues…

- What should teachers know and understand?
  e.g., student conceptions: The most salient ones. Students' conceptions may differ according to curricula, classroom practices…

- How should they learn?
  promising ways…

- When should they learn?
  during preservice education…
  during inservice professional development…

19

The Intuitive Rules Theory

Stavy and Tirosh

Introduction

The Intuitive Rule: ***more A – more B***

The Intuitive Rule: ***same A – same B***

Current studies

Science and mathematics education

Motive:

- Low achievements in science and mathematics
- Research on students' alternative conceptions (misconceptions) and reasoning
- Underlying assumption that understanding students' reasoning in mathematics will improve mathematics education

The Main Assertion:

Students' responses to mathematics and science tasks are determined in great part by:

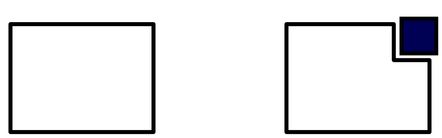
1. Specific, external characteristics of the task.

2. A small set of intuitive rules, and not necessarily by the tasks' content area.

Stavy, R., & Tirosh, D. (2000) *How students (mis-)understand science and mathematics, Teacher College Press.*

The Intuitive Rule: *more A -- more B*
Example I

A comparison of area and perimeter task:



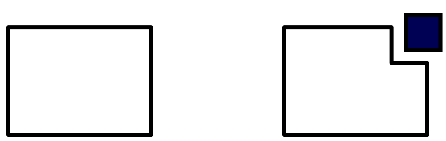
Rectangle   A small square is removed from the corner

The Intuitive Rule: *more A -- more B*
Example I

A comparison of area and perimeter task:

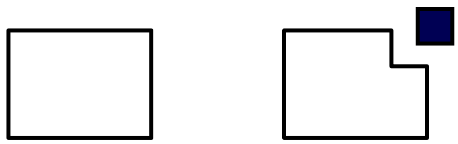
The Intuitive Rule: *more A -- more B*
Example I

A comparison of area and perimeter task:



The Intuitive Rule: *more A -- more B*
Example I

A comparison of area and perimeter task:



The Intuitive Rule: *more A -- more B*
Example I

A comparison of area and perimeter task:

The Intuitive Rule: *more A -- more B*
Example I

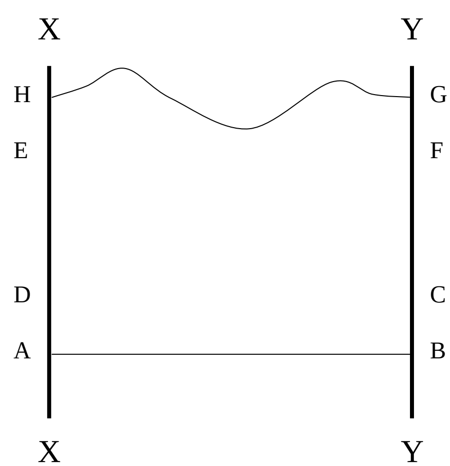
A comparison of area and perimeter task:

Shape I (rectangle)   Shape II (identical rectangle, from which a small square was removed → **polygon**)

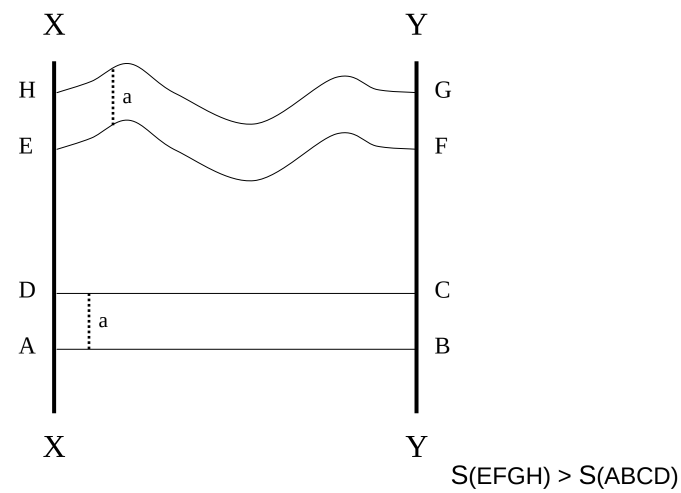
Q: Is the area of shape I larger than/smaller than/equal to the area of shape II **?**

Q: Is the perimeter of shape I larger than/smaller than/equal to the perimeter of shape II **?**

The Intuitive Rule: *more A -- more B*
Example II



The Intuitive Rule: *more A -- more B*
Example II



The Intuitive Rule: *more A -- more B*
Example II

- S(EFGH)> S(ABCD)

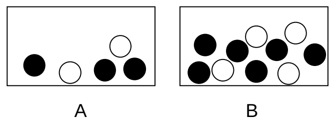
In this case, unlike the previous one:

A (the salient property) is the perimeter
B is the area.

- Fischbein, E. (1987) *Intuition in Science and Mathematics, D. Reidel*

The Intuitive Rule: *more A -- more B*
Example III

Two bags have black and white counters.



Which bag gives a better chance of picking a black Counter?

- Same chance
- Bag A
- Bag B
- Don't know

The Intuitive Rule: *more A -- more B*
Example IV

The size of a Lion's liver cell is:
smaller than
equal to
larger than
The size of a cat's liver cell?

Grade 7 (**74%)** incorrectly:
The lion's liver cell is larger, since the lion is "**bigger**" or "**stronger**" than the cat.

Babai et al. (2003) Unpublished

The Intuitive Rule: ***more A -- more B***

- Relationships between two objects that differ in a salient quantity A ($A_1 > A_2$) are presented
- Subjects are asked to compare the two objects with respect to quantity B (B_1 not bigger than B_2)

A substantial number of subjects responded incorrectly according to the rule ***more A*** *(the salient quantity) --* ***more B*** *(the quantity in question)*, claiming that $B_1 > B_2$

The Intuitive Rule: ***more A -- more B***

Responses of the type *More A -- more B* are observed in comparison tasks:

Classic Piagetian conservation tasks

Intensive quantities (density, temperature, concentration)

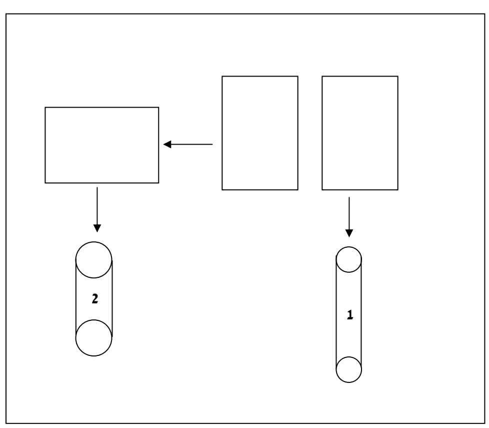
Other tasks (e.g., free fall, infinite sets, number of chromosomes).

The Intuitive Rule: *same A -- same B*
Example I

- Take two identical rectangular sheets of paper.
- Rotate one sheet by 90^0 (sheet 2)
- Fold each sheet. You get two cylinders.

Is the volume of cylinder 1 larger than/ equal to/ smaller than the volume of cylinder 2?

The Intuitive Rule: *same A -- same B*
Example I



The Intuitive Rule: *same A -- same B*
Example I

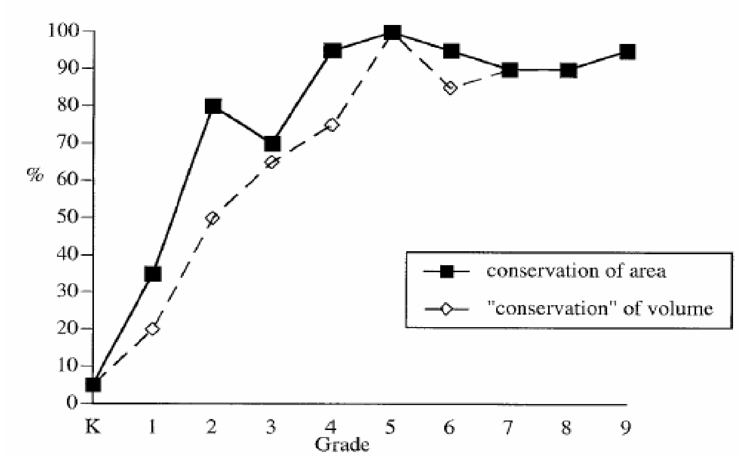
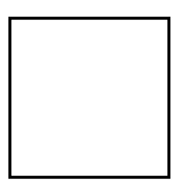


Figure 1. Distribution of equality judgment, by age, to the task of surface area and volume of two cylinders

The Intuitive Rule: *same A -- same B*
Example II

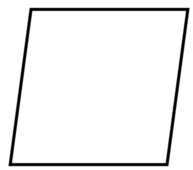
The sides of a square were moved without changing their length, to form a parallelogram



The Intuitive Rule: *same A -- same B*

Example II

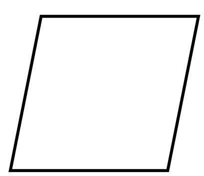
The sides of a square were moved without changing their length, to form a parallelogram



The Intuitive Rule: *same A -- same B*

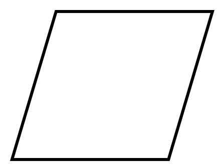
Example II

The sides of a square were moved without changing their length, to form a parallelogram



The Intuitive Rule: ***same A -- same B***
Example II

The sides of a square were moved without changing their length, to form a parallelogram

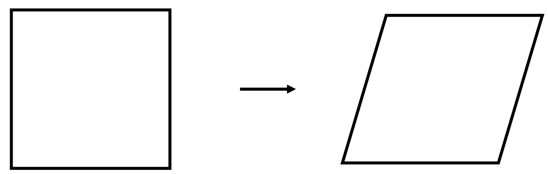


The Intuitive Rule: ***same A -- same B***
Example II

The shapes have the ***same perimeter.***

The derived parallelogram has a smaller area than the original square.

Subjects were asked to compare the area of the square and parallelogram.



The Intuitive Rule: *same A -- same B*
Example II

Grade	7	8	9
***Incorrect** (in line with rule)*	93	91	89
***Correct** (counter intuitive)*	2	4	5

The Intuitive Rule: *same A -- same B*

- Relationships between two objects that are equal in a salient quantity A (A_1=A_2) are presented
- Subjects are asked to compare the two objects with respect to B (B_1 not equal to B_2)
- Common response: ***same A*** *(the salient quantity)* -- ***same B*** *(the quantity in question)*.

Intuitive Rules

- Students' responses are determined in great part by the specific, external characteristics of the tasks, which trigger the intuitive rules

- So far, three intuitive rules were defined:

  more A -- more B

  same A -- same B

  everything can be divided endlessly

Intuitive Rules

Predictive power : when a certain task is described it is possible to predict how subjects will respond on the basis of external, specific features of the task.

Responses in line with the intuitive rules are often correct. However, these rules are in many cases not in line with concepts and reasoning in science and mathematics and lead to incorrect judgments.

Current Research

- **Impression -- the responses are:**
  self evident
  coercive
  immediate

  Therefore they were regard as
  Intuitive
- **However…**
  No empirical measurements of immediacy… until recently

Current Research

- **Immediacy**: A main characteristic of intuitive responses (e.g., Fischbein, 1987, Westcott, 1968)

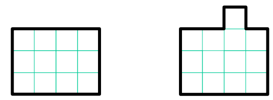
- Reaction time of students' responses to two types of responses:
  * Responses in line with the intuitive rules
  * Responses not in line with the intuitive rules

Experiment 1 - tasks

Two types of area and perimeter tasks:

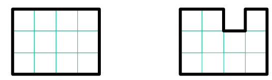
1. **Congruent**: correct response **is in line** with the intuitive rule.

   Larger area ($S_1<S_2$) - larger perimeter ($P_1<P_2$)



2. **Incongruent**: correct response **is not in line** with the intuitive rule.

   a) Larger area ($S_1>S_2$) - smaller perimeter ($P_1<P_2$)



   b) Larger area ($S_1>S_2$) - same perimeter ($P_1=P_2$)

One of the shapes was derived from the other by adding or removing of one grid square

Research design - procedure

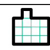
Participants: **22 students** (Grade 11 and 12)

Procedure:

- Computerized test, 48 comparison tasks (3 types)
- Two sessions: (perimeter, area).
- Subjects: asked to determine for each task:
  - left is larger
  - right is larger
  - both shapes are equal

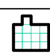
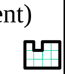
Participants were instructed to **answer correctly** and **as fast as they could**

Results

Nature of task		Area		Perimeter	
		RT for Correct Responses	% Error	RT for Correct Responses	% Error (RT for incorrect responses)
In line with the intuitive rule (congruent)		1227	1.4	1398	1.1
Not in line with the rule (incongruent)					
2a		1310	1.7	1634	0.9
2b		1188	1.4	1785	46.3 (1577)

Mean of median* RTs (in ms) of correct responses and error rates as a function of quantity and nature of task.

Results

Nature of task		Area		Perimeter	
		RT for Correct Responses	% Error	RT for Correct Responses	% Error (RT for incorrect responses)
In line with the intuitive rule (congruent)		1227	1.4	1398	1.1
Not in line with the rule (incongruent)					
2a		1310	1.7	1634	0.9
2b		1188	1.4	1785	46.3 (1577)
Mean		1242		1606	

Mean of median* RTs (in ms) of correct responses and error rates as a function of quantity and nature of task.

Conclusions

Area comparison is faster than perimeter

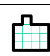
The area is indeed the salient property - A

larger area - larger perimeter

Perimeter comparison - **longer RTs** for incongruent situations (compared to congruent)

"**Slowing effect"** for correct response when conflicting area and perimeter inputs (perimeter incongruent situations)

Results

Property	Area		Perimeter	
Nature of task	RT for Correct Responses	% Error	RT for Correct Responses	% Error (RT for incorrect responses)
In line with the intuitive rule (congruent) 	1227	1.4	1398	1.1
Not in line with the rule (incongruent)				
2a 	1310	1.7	1634	0.9
2b 	1188	1.4	1785	46.3 (1577)

Mean of median* RTs (in ms) of correct responses and error rates as a function of quantity and nature of task.

Interestingly:

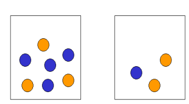
RTs for incorrect responses in line with the intuitive rule: 1577ms

shorter than

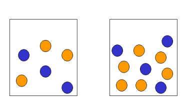
RTs for correct responses (counter to intuitive rule): 1785ms

Levyadun, T. (2003) MA thesis, Tel Aviv University

Probability: Examples of tasks



In line with the rule



Counter to the rule

Probability

Two types of tasks:

Congruent: correct response **in line** with the intuitive rule.

Larger number of blue counters ($N_1 > N_2$) - larger probability of picking a blue counter ($P_1 > P_2$)

Incongruent: correct response **not in line** with the intuitive rule.

Larger number of blue counters ($N_1 > N_2$) - smaller probability of picking a blue counter ($P_1 < P_2$)

Probability

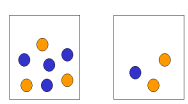
Participants: 80 high school students

Procedure: Computerized test with 40 probability tasks

Subjects were asked to determine for each task:

Larger probability of picking a blue counter:

Left bag
Right bag
Same



Participants were instructed to **answer correctly and as fast as they could**

Probability: Results

Tasks:	RT	% err
In-line with *Intuitive* rule	1620	8.7
Counter- *Intuitive*	2475	21.5

Conclusions

Responses in-line with the intuitive rule yielded faster, correct responses than counter intuitive responses.

Brecher, T (2003) MA Thesis, Tel Aviv University

Significance

- The Reaction Time results confirm the Intuitive Rules Theory.
- In science and mathematics there are many situations with contradictory information relating to the same phenomenon.
- Further study in the framework of the Intuitive rules will lead to better understanding of students' reasoning and will help dealing with common mistakes

---

- Is intuitive thinking different from logical thinking? In what ways?
- What makes a certain property salient at one situation and not at another?
- What are the implications for instruction?

Some Issues…

- What should teachers know and understand?
  e.g., student conceptions: The most salient ones. Students' conceptions may differ according to curricula, classroom practices…

- How should they learn?
  promising ways…

- When should they learn?
  during preservice education…
  during inservice professional development…

65

Thank you

66