Teachers' Knowledge of Students' Mathematical Learning: An Examination of a Commonly Held Assumption

Dina Tirosh Ruhama Even
Tel Aviv University Weizmann Institute of Science

It is widely accepted today that teachers should be aware of and knowledgeable about students' mathematical learning, as this significantly contribute to various aspects of the practice of teaching. We begin this paper by interpreting what one might mean by students' mathematical learning. Then we examine the validity of the assumption that teacher knowledge of students' mathematical learning is essential for good teaching in light of different theoretical perspectives. The third part describes teacher education programs that focus on different aspects of students' mathematical learning. We conclude by suggesting issues for further research.

WHAT COULD BE ENTAILED BY STUDENTS' MATHEMATICAL LEARNING?

We focus on three aspects of students' mathematical learning that have been at the center of researchers' attention during the last decades: student conceptions, different forms of knowledge and kinds of understanding, and classroom culture.

Student Conceptions

In the last decades many researchers have investigated students' mathematical ideas and conceptions as well as their development. Results of these studies show that learning mathematics is complex, takes time, and is often not straightforward (e.g., Bishop, Clements, Keitel, Kilpatrick, & Leung, 2003; English, 2002; Grouws, 1992; Gutiérrez & Boero, 2006; Nesher & Kilpatrick, 1990; Schoenfeld, Smith & Arcavi, 1993). Here we briefly describe three lines of that research: theory building, misconceptions and errors, and exploring what students know and can do.

1. Theory building. Prominent attempts in this direction include: (1) developing a comprehensive theory that describes how students learn specific mathematical domains or concepts (e.g., the van Hiele theory concerning geometry learning – van Hiele & van Hiele-Geldof, 1959); (2) constructing theories that are not specific to learning in a certain mathematical domain but rather suggest general principles for learning mathematics (e.g., the acquisition of mathematical concepts – Davis, 1975; Dubinsky, 1991; Gray & Tall, 1994; Sfard, 1991).

2. Misconceptions and errors. A much more prominent line of research in the field of mathematics education is the study of errors. While the previous avenue of research focuses on general aspects of students' learning of mathematics, here, researchers usually focus on specific "local" concepts. Some researchers engage in describing in detail errors that students make in specific topics (e.g., Gutiérrez & Boero, 2006; Nesher & Kilpatrick, 1990). Others explore additional dimensions including the evolution of misconceptions with age and instruction (e.g., Fischbein & Schnarch,1997; Hershkowitz,1987; Vosniadou & Verschaffel, 2004). Several theoretical frameworks attempt to describe general, underlying sources of students' incorrect responses. Here we briefly describe one such theory: the Intuitive Rules Theory (Stavy & Tirosh, 2000).

The essential claim of the Intuitive Rule Theory is that irrelevant, external features of mathematical and scientific tasks often determine responses to them. For instance, students' responses to comparison tasks embedded in different topics are often of the type *more A – more B* (Stavy & Tirosh, 2000). One example relates to right angles. Studies have shown that when children in grades K to 4 are presented with two right angels, drawn with the same length of arms, the equality of the angles appear to them as self-evident. However, when the same children are asked to compare two right angles, one drawn with longer arms than the other, they claim that the angle with the longer arms is larger. This judgment exemplifies the effect of the intuitive rule *more A* (the perceived length of the arms)– *more B* (the size of the angles). This and other rules bear the characteristics of intuitive thinking: They appear self-evident, are used with great confidence, and alternative responses are excluded as unacceptable (Fischbein, 1987). The Intuitive Rules Theory explains numerous incorrect responses and has a strong predictive power.

3. Exploring what students know and can do. A newer trend is the focus on what students know and can do, highlighting the continuity in knowledge between novices and masters (e.g., Lamon, 2006; Smith, diSessa, & Roschelle, 1993; Streefland, 1993). For example, Smith, diSessa, and Roschelle (1993) showed fundamental similarities in characteristics of masters' and novices' knowledge about fractions: Both groups tended to construct strategies that were tailored to solving specific classes of problems instead of using the more general school-taught strategies.

Different Forms of Knowledge and Kinds of Understanding

The notions "knowledge" and "understanding" are multi-dimensional. Different forms of knowledge and various kinds of understanding are described in the mathematics education literature (e.g., instrumental, relational, conceptual, procedural, implicit, explicit, elementary, advanced, algorithmic, formal, intuitive, visual, situated, knowing that, knowing how, knowing why, knowing to). Here we present a brief description of several of these forms, portraying some major themes.

1. Instrumental understanding and relational understanding: A dichotomy or a continuum? In his classic article, Skemp (1978) presented relational understanding as knowing both what to do and why, and instrumental understanding as “rules without reasons” (p. 9). Skemp argued that these two kinds of knowledge should be regarded as *different kinds of mathematics*. He presented several, severe problems that could occur when pupils whose goal is to understand instrumentally are taught by a teacher who wants them to understand relationally, or vice versa. Skemp’s work contributes significantly to the long-standing debate on the relative importance of computational skill versus mathematical understanding and to further investigations and discussions on this issue (e.g., Hiebert & Carpenter, 1992; Nesher, 1986; Resnick and Ford, 1981).

2. Algorithmic, formal and intuitive dimensions of mathematics: Interactions and inconsistencies. Fischbein (1993) suggested that any mathematical activity requires the use of three dimensions of mathematic knowledge: algorithmic (procedures for solving and their theoretical justifications), formal (axioms, definitions, theorems, and proofs) and intuitive (common mental models, ideas and beliefs about mathematical entities). Fischbein argued that these three dimensions of knowledge overlap considerably. Ideally, the dimensions should operate in harmony in the processes of concept acquisition, understanding and problem solving. In reality, though, there are serious inconsistencies among students' algorithmic, intuitive and formal knowledge. These inconsistencies are expressed as misconceptions and cognitive obstacles.

3. Knowing-about and knowing-to. A rather frustrating phenomenon, often described by both researchers and teachers, is that students, who are known to have all the necessary knowledge to solve a problem, are unable to employ this knowledge to solve unfamiliar problems. Attempting to explain this phenomenon, Mason and Spence (1999) define a special form of knowing: *knowing-to act in the moment*. They suggest that this knowledge

enables people to act creatively rather than merely react to stimuli with habituated behavior. Mason and Spence claim that *knowing-to* requires sensitivity to situational features and some degree of awareness of the moment, so that relevant knowledge is accessed when appropriate. They suggest that *knowing-to* is the type of knowing students need to engage in problem solving in novel contexts where solutions are non-routine.

Classroom Culture

An important issue that has received the attention of the mathematics education community in recent years is classroom culture (Even & Schwarz, 2003; Lerman, 2006). This new focus signals a shift from examining human mental functioning in isolation to considering cultural, institutional and historical factors.

Several mathematics educators (e.g., Ball, 1991; Chazan, 2000; Cobb, Stephan, McClain, & Gravemeijer, 2001; Lampert, 1990; Schoenfeld, 1994; Yackel, 2001) have attempted to support the development of a different mathematical culture in the classroom. A main characteristics of this renewal is the alteration of traditional roles and responsibilities of teacher and students in classroom discourse. These researchers and others (e.g., Goos, 2004; Wood, Williams, & McNeal, 2006) investigate mathematics learning and knowing in such classrooms. They document the social and the sociomathematical norms that sustain classroom cultures (Yackel & Cobb, 1996). The tendency to study the nature mathematical classroom cultures, and how they might be developed, is growing. Consequently, issues such as the notion of identity that has been traditionally studied in mathematics education by addressing the learners' self concept and self efficacy, are currently also studied with reference to socio-cultural notions (e.g., Lerman, 2006).

SHOULD TEACHERS BE AWARE OF STUDENTS' MATHEMATICAL LEARNING?

At the beginning of this paper we stated that it is widely accepted that knowledge and understanding of students' mathematical learning is important for teaching. In this section, for each of the three aspects of students’ mathematics learning discussed in the previous section, we examine the validity of this assumption in light of three dominant learning perspectives: behaviorism, constructivism, and situationism.

Knowing about Student Conceptions

Behaviorism views learning as the process in which associations and skills are acquired. A basic assumption is that any use of a wrong association tends to strengthen it. Therefore, it is essential to prevent students from making mistakes or from being exposed to errors made by their peers. Behaviorists state explicitly that it is impossible for anyone (including teachers) to know what goes on in the students' mind. They direct teachers toward determining the correctness of the students’ responses, not the students' conceptions.

According to constructivism, children’s knowledge differs not only quantitatively but also qualitatively from that of the adult. A basic assumption of constructivism is that knowledge is not communicated but constructed by unique individuals. When teaching mathematics, the teacher should attend to students’ thinking, form an adequate model of students' ways of viewing an idea, and then construct a tentative path along which students may move to construct a mathematical idea. Accordingly, the very essence of constructivism is to know and understand student conceptions.

The situationist perspective focuses on the kinds of social engagements that provide the proper context for learning to take place. Learning is perceived as a process that takes place in a participative framework, not in an isolated individual mind. The learner does not gain a discrete body of abstract knowledge, which s/he will then apply in other contexts. Rather, knowing is viewed as the practices of a community and the abilities of individuals to participate in those practices; learning is the strengthening of those practices and participatory abilities. Thus, the situationist perspective attends to students’ ability to participate in shared mathematical activities.

Knowing about Forms of Knowledge

Behaviorists view knowledge as an organized accumulation of facts, skills and procedures. Consequently, these are the types of knowledge that teachers are apt to emphasize in instruction. The constructivist perspective emphasizes the development of different forms of knowledge such as conceptual knowledge, problem solving strategies, and meta-cognitive abilities. Consequently, teachers should be knowledgeable about different forms of knowledge. Knowing-to is a central feature of participation. However, since the situationist perspective does not concentrate on knowledge per se, on this approach, knowing about different forms of knowledge may well be considered irrelevant for teachers. What might instead be important is attention to participation in activities, which involve the use of different forms of knowledge.

Knowing about Classroom Culture

Learning environments designed according to behaviorist principles are organized so that teachers efficiently transmit facts and procedural knowledge. Usually, the teacher presents correct procedures and provides opportunities for practice. The focus is on the individual student and the classroom is viewed as a collection of individual students. Constructivist learning environments are designed to provide students with opportunities to construct conceptual understanding and to foster problem-solving and reasoning abilities. Constructivism[1] focuses on the individual student, not on building a community of learners. In the situationist perspective, by contrast, the classroom culture is the essence. Teachers represent the community of practice, exemplify valued practices, encourage the development of desired norms, and guide students as they become increasingly competent practitioners.

Navigating between Perspectives

It is clear that each learning perspective approaches the teacher's knowledge about student learning differently. We join Sfard (1998) in arguing that choosing and being completely loyal to one learning perspective is counter-productive in educational practice. We believe that understanding student conceptions, both those documented in the research literature and those known from experience, would assist teachers to adjust instruction to where their students are in their mathematical understanding. Also, it is important for teachers to be aware that knowing mathematics cannot be reduced to one simple form of knowledge. Furthermore, teachers should be aware that classroom culture is inseparable from learning mathematics, as learning always occurs in a specific sociocultural environment. It is essential for teachers to understand the interrelations between classroom norms and mathematics learning - on the basis of this understanding they can then construct and maintain an appropriate learning environment.

In this last section of the paper we adapt the claim that knowledge about students' mathematical learning is valuable for teaching. We present several pre- and in-service teacher education programs that focus on different aspects of this knowledge.

---

[1] Social constructivism does take account of the social aspect of learning. Yet, it centers on the individual learner in a social context and not on the class as a community.

TEACHER EDUCATION: WHAT AND HOW

Research and professional rhetoric recommend that attention be paid to students' mathematics learning in teacher education and professional development programs (e.g., Cobb & McClain, 1999; Even, 1999; Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996; Llinares & Krainer, 2006; Rhine, 1998; Scherer & Steinbring, 2006; Simon & Schifter, 1991; Tirosh & Graeber, 2003). This recommendation is based on the view that awareness to and understanding of students' mathematics learning are central to good teaching and that these, moreover, do not come about automatically. Here, we do not attempt to provide a survey of programs that adopt such practice. Rather, we limit ourselves to discussing what it might mean for teacher education to focus on students' mathematical thinking. We organize our discussion around the three aspects of student mathematical learning which have been serving as our foci throughout this paper: student conceptions, different forms of knowledge, and classroom culture.

Educating about Student Conceptions

Many teacher education programs center on developing teachers' knowledge about students' mathematical conceptions. Some concentrate on teaching specific theories and models of students' mathematical thinking. Others aim at developing awareness that students often think differently about mathematical concepts than what might be expected. A pioneering, successful project entitled Cognitively Guided Instruction (CGI) has focused on enabling inservice elementary school teachers to understand their students' thinking by using a specific research-based model of children's mathematical thinking (Fennema et al., 1996). In contrast with CGI, the Manor Program for the development of a professional group of secondary school mathematics teacher-leaders and in-service teacher educators (Even, 1999, 2005a) does not provide the participants with explicit research-based models of children's thinking in specific mathematical topics. Rather, similar to the Integrating Mathematics Assessment (Rhine, 1998) and the Mathematics Classroom Situations (Markovits & Even, 1999) approaches, the aim is for the participants to become acquainted with research-based key features of student thinking in different mathematical topics (e.g., cognitive development and aspects of mathematical thinking in algebra, analysis, geometry and probability). By conducting a mini-study with real students the Manor participants learn that what they thought they knew about their students was not necessarily a good representation of the students' knowledge and abilities (similar results are reported by Lerman, 1990 and by D'Ambrosio and Campos, 1992). Depending on their background and the specific project they chose to work on, some Manor participants learned that, contrary to expectations, their students could actually deal with sophisticated mathematical ideas that seemed too difficult,. Others found that even well planned teaching might not produce the kind of learning they expected.

Educating about Different Forms of Knowledge

A thorough review of preservice teacher education programs and inservice professional development projects suggests that these programs and projects usually do not declare learning about various forms of mathematics knowledge as their principal aims. Many, however, state that designing opportunities for teachers to develop deeper understandings of the mathematics they are to teach and enhancing teachers' understanding of their students' mathematical thinking are two of their main aims. Thus, although it is not listed as an explicit aim of such programs, highlighting learning about various forms of mathematics knowledge is an implicit goal of many of them. Here we shall briefly describe a one-year preservice elementary school teacher program, Students' Thinking About Rationals (STAR), which concentrates on participants' subject matter knowledge and pedagogical content knowledge of rational numbers (Tirosh, 2000). One aim of this program is to familiarize prospective teachers with Fischbein's framework of the three basic dimensions of mathematics knowledge: algorithmic, formal, and intuitive. We believe that this framework could support teachers in their attempts to foresee, interpret, explain and make sense of students' mathematics learning.

Fischbein's framework was frequently used in the course. We present here one example relating to division of fractions. Participants were requested to (a) calculate four division expressions, (b) list common mistakes seventh grade students may make after completing their studies on fractions, and (c) describe possible sources for each of these mistakes. One of the expressions was $\frac{1}{4} \div \frac{1}{2}$. At the beginning of the course all participants calculated this expression correctly. Most of them argued that the (only) common mistake students would make is $\frac{1}{4} \div \frac{1}{2} = 2$ and that this mistake would originate from a bug in the algorithm (e.g., $\frac{1}{4} \div \frac{1}{2} = \frac{4}{1} \cdot \frac{1}{2} = 2$). During the course, Fischbein's framework was used to exemplify that such a response could derive from the commonly held intuitive belief that in division, the dividend should always be greater than the divisor (and therefore $\frac{1}{4} \div \frac{1}{2} = \frac{1}{2} \div \frac{1}{4} = 2$), from inadequate formal knowledge (e.g., division is commutative and therefore $\frac{1}{4} \div \frac{1}{2} = \frac{1}{2} \div \frac{1}{4} = 2$) or from other sources. By the end of the course most participants were acquainted with Fischbein's framework and used it to guide their attempts to describe common incorrect responses.

Educating about Classroom Culture

As the focus on socio-cultural aspects is relatively new among mathematics educators, it is only natural that most of the emphasis is currently centered on examining socio-cultural aspects of student learning and not yet on educating teachers about it. In their pioneering work in this direction with inservice elementary school teachers, Cobb and McClain (1999) use episodes from classrooms to serve as a basis for conversations with teachers about the role of the teacher in supporting the development of sociomathematical norms. Similarly, Lampert and Ball (1999) created multimedia records of practice. Preservice teachers explore the records of practice in the multimedia environment aiming to identify items that exemplify key elements of the culture of the classroom, formulating conjectures about the teacher's role in establishing and maintaining these elements of classroom culture. In doing so, Lampert and Ball drew attention to what constitutes classroom culture and how it can be developed in a classroom as content to be learned by prospective teachers.

LOOKING TO THE FUTURE

In this paper we discussed teachers' knowledge and understanding of students' mathematical learning. Three main relevant issues are:

- What should teachers know and understand?
- How should they learn?
- When should they learn?

In the preceding sections we focused on the first two questions (*What?* and *How?*) in light of the information provided by the research literature. Much less is known about the third question, *When?* (e.g., during preservice education? during inservice professional development?). We approached the *What?* and *How?* questions by referring to three aspects: (1) student conceptions, (2) different forms of knowledge, and (3) classroom culture.

There are other issues that need to be examined such as, *What do teachers need to know about these aspects?* and *What are promising ways for teacher learning about them?* For example, regarding student conceptions, a spontaneous solution may be to choose the most salient ones. However, students' conceptions may differ according to the curricula they study, the classroom practices they experience, and other factors. The extent to which mathematical ways of thinking and difficulties are embedded in a particular approach to learning and teaching still needs to be studied.

A similar issue emerges in relation to educating teachers about forms of knowledge. Currently, there is no one single theoretical framework that is widely accepted by the mathematics education community. Consequently, decisions should be made regarding which and how many frameworks will be used.

With respect to research on classroom culture, we feel that the literature does not provide enough critical analyses of problematic aspects, of advantages and disadvantages of adapting the current advocated classroom culture. Missing are analyses that take into account the complexity of actual mathematics instruction which needs to consider various (and sometimes conflicting) factors, facets and circumstances. Even if we adopt the vision of a desired classroom culture as advocated today in reform documents we are still faced with questions concerning *How?* Is it necessary for teachers to experience a desired classroom culture as learners? Is it sufficient? Do they need to observe such classrooms? Is it enough? Do they need to actually experience teaching in such classrooms as student teachers?

Understanding students' thinking is a problematic notion, because, as Wood (2004) points out, "teachers and teacher educators…may use the same words, but may talk past each other as the meanings held by each are quite different" (p. 173). Recent studies (Even, 2005b; Wallach & Even, 2005) indicate that there are often discrepancies between what students say and do, and what teachers understand, suggesting that teacher interpretation of students' understanding, knowledge and learning of mathematics draws on a rich knowledge base of understandings, beliefs, and attitudes. Consequently, the process whereby a teacher makes sense of students' understanding involves ambiguity and difficulties. As we saw earlier, current teacher education programs aim to raise teachers' awareness of the importance of understanding their students' mathematics conceptions and to develop teachers' knowledge about different ways in which students think and reason mathematically. The recent findings (Even, 2005b; Wallach & Even, 2005) suggest that in teacher education it is also important to address the issue that what the teacher understands could be different from what students are saying or doing.

Finally, although we raised many issues regarding teacher knowledge and understanding of students' mathematical learning that still need to be explored, we would like to stress that our research community has made considerable progress with respect to this issue in the last decades. This research has advanced our understanding of the complex nature of teacher knowledge in general, of teacher knowledge and understanding about student mathematical learning, in particular, and of the interrelations of this kind of knowledge with instructional practice. We look forward to seeing what exciting research the next decades will bring.

REFERENCES

Ball, D. L. (1991). What's all this talk about "discourse"? Arithmetic Teacher, 39(3), 44-48.

Bishop, A. J., Clements, K., Keitel, C., Kilpatrick, J., & Leung, F. K. S. (Eds.) (2003). Second international handbook of mathematics education. Dordrecht, Netherlands: Kluwer.

Chazan, D. (2000). Beyond formulas in mathematics and teaching: Dynamics of the high school algebra classroom. New York: Teachers College Press.

Cobb, P., & McClain, K. (1999). Supporting teachers' learning in social and institutional context. In F. Lin (Ed.), Proceedings of the 1999 International Conference on Mathematics Teacher Education (pp. 7-76). Taipei, Taiwan: National Taiwan Normal University.

Cobb, P., Stephan, M., McClain, K., & Gravemeijer, K. (2001). Participating in classroom mathematical practices. The Journal of the Learning Sciences, 10(1&2), 113-163.

d'Ambrosio, B. S., & Campos, T. M. M. (1992). Pre-service teachers' representations of children's understanding of mathematical concepts: Conflicts and conflict resolution. Educational Studies in Mathematics, 23, 213-230.

Davis, R. B. (1975). Cognitive processes involved in solving simple algebraic equations. Journal of Children's Mathematical Behavior, 1 (3), 7-35.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. Tall (Ed.), Advanced mathematical thinking (pp. 95-123). Dordrecht: Kluwer Academic Publishers.

English, L. (Ed.) (2002). Handbook of international research in mathematics education. Mahwah, NJ: Laurence Erlbaum.

Even, R. (1999). Integrating academic and practical knowledge in a teacher leaders' development program. Educational Studies in Mathematics, 38, 235-252.

Even, R. (2005a). Integrating knowledge and practice at MANOR in the development of providers of professional development for teachers. Journal of Mathematics Teacher Education, 8(4), 343-357.

Even, R. (2005b). Using assessment to inform instructional decisions: How hard can it be? Mathematics Educational Research Journal, 17 (3), 51-67.

Even, R., & Schwarz, B. B. (2003). Implications of competing interpretations of practice to research and theory in mathematics education. Educational Studies in Mathematics, 54 (2), 283-313.

Fennema, E., Carpenter, T., Franke, M., Levi, L., Jacobs, V., & Empson, S. (1996). A longitudinal study of learning to use children's thinking in mathematics instruction. Journal for Research in Mathematics Education, 27(4), 403-434.

Fischbein, E. (1987). Intuition in science and mathematics, Dordrecht, The Netherlands: Reidel.

Fischbein, E. (1993). The interaction between the formal, the algorithmic and the intuitive components in a mathematical activity. In R. Biehler, R. Scholz, R. Straser and B. Winkelmann (Eds.), Didactics of mathematics as a scientific discipline (pp. 231-245). Dordrecht, The Netherlands: Kluwer.

Fischbein, E., & Schnarch, D. (1997). The evolution with age of probabilistic, intuitively-based misconceptions. Journal for Research in Mathematics Education, 28, 96-105.

Goos, M. (2004). Learning mathematics in a classroom community of inquiry. Journal for Research in Mathematics Education, 35(4), 258-291.

Gray, E., & Tall, D. (1994). Duality, ambiguity and flexibility: A proceptual view of simple arithmetic. Journal for Research in Mathematics Education, 26 (2), 115-141.

Grouws, D. A. (Ed.) (1992). Handbook of research on mathematics teaching and learning. NY: Macmillan.

Gutiérrez, A., & Boero, P. (Ed.) (2006). Handbook of research on PME: Past, present and future. Rotterdam, the Netherlands: Sense.

Hershkowitz R. (1987). The acquisition of concepts and misconceptions in basic geometry--Or when "a little learning is a dangerous thing." In J. D. Novak (Ed.), Proceedings of the second international seminar on misconceptions and educational strategies in science and mathematics (Vol. 3, pp. 238-251). Itacha, NY: Cornell University.

Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grouws, (Ed.), Handbook of research on mathematics teaching and learning (pp. 65-97). New York: Macmillan.

Lamon, S. J. (2006). Teaching fractions and ratios for understanding: essential content knowledge and instructional strategies for teachers (2nd edition). Hillsdale, New Jersey: Lawrence Erlbaum.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. American Educational Research Journal, 27, 29-63.

Lampert, M. & Ball, D.L. (1998). Teaching, multimedia, and mathematics: Investigations of real practice. New-York: Teachers College.

Lerman, S. (1990). The role of research in the practice of mathematics education, For the Learning of Mathematics, 10(2), 25-28.

Lerman, S. (2006). Socio-cultural research in PME. In A. Gutiérrez & P. Boero (Ed.) (2006). Handbook of research on PME: Past, present and future (pp. 347-366). Rotterdam, the Netherlands: Sense.

Llinares, S., & Krainer, K. (2006). Mathematics (students) teachers and teacher educators as learners. In A. Gutiérrez & P. Boero (Ed.) (2006). Handbook of research on PME: Past, present and future (pp. 347-366). Rotterdam, the Netherlands: Sense.

Markovits, Z., & Even, R. (1999). The decimal point situation: A close look at the use of mathematics-classroom-situations in teacher education. Teaching and Teacher Education, 15(6), 653-665.

Mason, J., & Spence, M. (1999). Beyond mere knowledge of mathematics: The importance of knowing-to act in the moment. Educational Studies in Mathematics, 38, 135-161.

Nesher, P. (1986). Are mathematical understanding and algorithmic performance related? For the Learning of Mathematics, 6 (3), 2-9.

Nesher, P., & Kilpatrick, J. (Eds.) (1990). Mathematics and cognition. Cambridge, MA: Cambridge University Press.

Resnick, L. B., & Ford, W. W. (1981). The psychology of mathematics for instruction. Hillsdale, NJ: Erlbaum.

Rhine, S. (1998). The role of research and teachers' knowledge base in professional development. Educational Researcher, 27(5), 27-31.

Scherer P., & Steinbring, H. (2006). Noticing children's learning processes – teachers jointly reflect on their own classroom interaction for improving mathematics teaching. Journal for Mathematics Teacher Education, 9, 157-185.

Schoenfeld, A. H. (1994). Reflections on doing and teaching mathematics. In A. H. Schoenfeld (Ed.), Mathematical thinking and problem solving (pp. 53-70). Hillsdale, NJ: Erlbaum.

Schoenfeld, A. H., Smith, J. P. III, & Arcavi, A. (1993). Learning: The microgenetic analysis of one student's evolving understanding of a complex subject matter domain. In R. Glaser (Ed.), Advances in instructional psychology (Vol. 4, pp. 55-175). NJ: Erlbaum.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. Educational Studies in Mathematics, 22 (1), 1-36.

Sfard, A. (1998). On two metaphors for learning and the dangers of choosing just one. Educational Researcher, 27(2), 4-13.

Simon, M., & Schifter, D. (1991). Towards a constructivist perspective: An intervention study of mathematics teacher development. Educational Studies in Mathematics, 22, 309-331.

Skemp, R. R. (1978). Relational understanding and instrumental understanding. Arithmetic Teacher, 26 (3), 9-15.

Smith, J. P. III, diSessa, A. A., & Roschelle, J. (1993). Misconceptions reconceived: a constructivist analysis of knowledge in transition. Journal of the Learning Sciences, 3, 115-163.

Stavy, R., & Tirosh, D. (2000). How students (mis)-understand science and mathematics. New York, NY: Teachers College Press.

Streefland, L. (1993). Fractions: A realistic approach. In T. R. Carpenter, E. Fennema, & T. A. Romberg (Eds.). Rational numbers: An integration of research (pp. 289-325). Hillsdale, New Jersey: Lawrence Erlbaum.

Tirosh, D. (2000). Enhancing prospective teachers' knowledge of children's conceptions: The case of division of fractions. Journal for Research in Mathematics Education, 31, 5-25.

Tirosh, D., & Graeber, A. (2003). Challenging and changing mathematics teaching classroom practices. In A. J. Bishop, M. A. Clements, C. Keitel, J. Kilpatrick, & F. K. S. Leung (Eds.), Second international handbook of mathematics education (pp. 643-688). The Netherlands: Kluwer.

van Hiele P. M., & van Hiele-Geldfof, D. (1959). A method of initiation into geometry. In H. Freudenthal (Ed.), Report on methods of initiation into geometry. Groningen: Walters.

Vosniadou, S,, & Verschaffel, L. (2004). Extending the conceptual change approach to mathematics learning and teaching. Learning & Instruction, 14, 445-451.

Wallach, T., & Even, R. (2005). Hearing students: The complexity of understanding what they are saying, showing, and doing. Journal of Mathematics Teacher Education, 8(5), 393-417.

Wood, T. (2004). In mathematics classes what do students' think? Journal of Mathematics Teacher Education, 7, 173-174.

Wood, T., Williams, G., & McNeal, B. (2006). Children's mathematical thinking in different classroom cultures. Journal for Research in Mathematics Education, 37(3), 222-255.

Yackel, E. (2001). Explanation, justification and argumentation in mathematics classrooms. In M. van den Heuvel-Panhuizen (Ed.), Proceedings of the 25th International Conference for PME (Vol 1, pp. 9-24). Utrecht, the Netherlands.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. Journal for Research in Mathematics Education, 27, 458-477.