Changed Views on Mathematical Knowledge in the Course of Didactical Theory Development – Independent Corpus of Scientific Knowledge or Result of Social Constructions?

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Abstract. The paper tries to show how the German didactical tradition has evolved in response to new theoretical ideas and new – empirical – research approaches in mathematics education. First, the classical mathematical didactics, notably ›stoffdidaktik‹ as a specific German tradition is described. The critiques raised against ›stoffdidaktik‹ concepts (for example, forms of ›progressive mathematisation‹, ›actively discovering learning processes‹ and ›guided reinvention‹ (cf. Freudenthal, Wittmann)) changed the basic views on the roles that ›mathematical knowledge‹, ›teacher‹ and ›student‹ have to play in teaching-learning processes; this conceptual change was supported by empirical studies on the professional knowledge and activities of mathematics teachers (for example, empirical studies of teacher thinking (cf. Bromme)) and of students conceptions and misconceptions (for example, psychological research on students' mathematical thinking). With the interpretative empirical research on everyday mathematical teaching-learning situations (for example, the work of the research group around Bauersfeld) a new research paradigm for mathematics education was constituted: the cultural system of mathematical interaction (for instance, in the classroom) between teacher and students. The interpretation of (school-) mathematics as a *theoretical* and *dynamic* knowledge corpus was the basis of a new theorisation of classroom knowledge building (for example, Steinbring). This theoretical view permitted to integrate mathematics in a new way into the social teaching-learning system: this knowledge is interactively constructed on the basis of its epistemological constraints.

1) Introduction

This contribution tries in an exemplary way by looking at the case of the development of important tendencies in mathematics education (Mathematikdidaktik) in Germany (partly in connection with international aspects) to investigate the clarification process of the central objects of mathematics education research and to analyze the important role the content matter ›mathematics‹ plays for teaching and learning processes.

›Mathematics *learning*‹ as an object of didactical considerations has been regarded instantly as the triad ›Learner – Teacher – Learning/Teaching-Content‹ at all times. In pedagogics, these three elements are labeled as the ›didactical triangle‹ since Friedrich Herbart (1776-1848) (see Peterßen 2001, p.140, and Künzli 2000, p. 48/49). According to Herbart »… education within instruction does not [take place] in the immediate relationship between educator and pupil, but educator and pupil [enter] into an *indirect relationship* to each other. Between them stand the instruction objects ([3] One can suppose that this is where the famous didactical triangle originates)« (Peterßen 2001, p. 140).

In mathematics education (in Germany) the didactical triangle has a long tradition. The vertices for mathematics education represent: (1) the mathematical knowledge, (2) the student and (3) the teacher (cf. Steinbring 1998a):

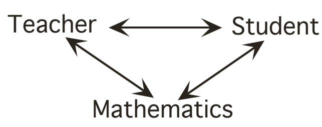

Fig. 1: The didactical triangle.

The schema of the didactical triangle with its three elements shall be used as a kind of ›test instrument‹ for the following considerations and analyses. By using this triangle, the following orientating questions will be asked for the elaboration of the changes and developments in mathematics education:

(A) Which explicit and implicit (unconscious) concepts and role descriptions exist about the three ›elements‹: mathematics, teacher and students?

(B) Which explicit and implicit (unconscious) concepts and role descriptions exist about the relationships or interactions between the three ›elements‹: mathematics, teacher and students?

(C) What is seen explicitly or implicitly (unconsciously) as the central and crucial means (among the three ›elements‹: mathematics, teacher and students?) of positively influencing and improving the learning process?

These questions, in combination with insights and answers are to help obtaining a more and more differentiated picture in the historical development of which research concepts and objects in mathematics education were predominant and how in this development the role and nature of (school-) mathematical knowledge was changed and redesigned.

2. The ›stoffdidaktic‹ elaboration of mathematical knowledge as an essential influence factor for teaching and learning processes

Until the middle of the sixties, in Germany there were mainly such didactical works and analyses in the foreground, which almost exclusively concentrated on the school-mathematical knowledge, its didactical elementarisation and on subject matter aspects. These works are essentially linked to mathematics as a pre-given learning and instruction content and specific features of a genuinely mathematics education research approach do not yet become noticeable.

Especially within the German-speaking countries, this didactical research paradigm has established as so-called ›stoffdidaktik‹ (Content based Didactics). »… in ›stoffdidaktik‹ dominates a too simple model to solve didactical questions and research problems. [It] acts on the assumption that the mathematical knowledge – researched and developed in the academic discipline – is essentially unchanged and absolute.

… Though ›stoffdidaktik‹ in the meantime notices the understanding problems of learning students, and accordingly it is a specific proceeding to prepare the pre-given mathematical disciplinary knowledge for instruction to a mathematical *content*, to elementarise it and to pack it methodically; yet it remains unchallenged that mathematical knowledge on principle represents a finished product, and that the teaching-learning-process is organizable linearly emanating from the content over the teacher to the students' heads and can ultimately be dominated and affected in all steps by the mathematics educators« (Steinbring1997, p. 67, see also Steinbring 1998a and Steinbring 1998b, p. 161/2).

With the abbreviated label ›stoffdidaktik‹, the representatives of this direction meant especially the ›didactically oriented content analysis‹. »The research complex of didactically oriented content analysis (Sachanalysen) has lately engaged mathematics education in the Federal Republic of Germany in a particular way. ... The research methods of this area are identical with those of mathematics, so that occasionally for outsiders the impression emerged that here, mathematics (particularly elementary mathematics) and not mathematics education were conducted. ... The goal of the ›didactically oriented content analysis‹ which essentially work with mathematical methods is giving a better foundation for the formulation of the content related learning goals and for the development, the definition and the use of a differentiated methodical instrumentarium.« (Griesel 1974, p. 118). In the centre of a genuinely mathematics education research of this time stood ›didactically oriented content analysis‹. »What does the progress in mathematics education depend upon? 1. Upon the development status of the analysis of the content, the methods and the application of mathematics. 2. Upon didactical ideas and incursions, which allow to better or at all attend a subject area within instruction.« (Griesel 1971, p. 7). Griesel names four further influence factors (general and statistically secured instruction experience, insights into the mathematical learning process, development-psychological and sociological conditions); yet the didactical work on the ›content‹ is the most important.

In a critical comparison between (German) ›didactically oriented content analysis‹ and (French) ›ingénieri didactique‹, R. Strässer (1994) states that ›stoffdidaktik‹ ultimately pursued the goal of elaborating school-mathematical subject areas – similar to mathematical areas in Bourbakism – in a logically consistent way and built upon unambiguous foundations. As an example, Strässer cites from the foreword to the two-volume book by G. Holland ›Geometry for Teachers and Students‹ (1974 / 77): »This book arguably offers the reader a complete axiomatic composition of the Euclidian geometry of the plane, which in its system of concepts as well as in the choice and organisation of the geometrical contents orientates itself as much as possible at the contemporary geometry instruction at school« (Holland 1974, p. 7).

An archetype for ›stoffdidaktik‹ was *uniform mathematics*, as it was exemplarily given by Bourbaki and then by the so-called New Mathematics. In this sense, mathematics is *the uniform* mathematics, in spite of the specialization and differentiation in its many subdisciplines. Bourbaki describes his thesis of uniform scientific mathematics in an exemplary and paradigmatic way. This Bourbakist viewpoint has also substantially influenced the characterization of the structure of school-mathematical knowledge. In his architecture of mathematics, Bourbaki writes:

»... it is possible to ask whether this luxuriant proliferation [of mathematics] is the growth of a vigorously developing organism which gains more cohesion and unity from its daily growth, or whether on the contrary it is nothing but the external sign of a tendency toward more and more rapid crumbling due to the very nature of mathematics, and whether mathematics is not in the process of becoming a Tower of Babel of autonomous disciplines, isolated from one another both in their goals and in their methods, and even in their language. In a word, is the mathematics of today singular or plural?« (Bourbaki 1971, p.24).

He answers the question himself, in that he works out the core of *uniform* mathematics by the axiomatic method and by types of structure. »As a consequence of the perceptible unity and coherence of scientific mathematics, mathematical knowledge as a whole, especially the mathematical content taught and learned at different school levels, is also regarded as uniform knowledge, equal and invariant for all intents and purposes. The image of uniform scientific research mathematics determines the idea of mathematical knowledge in its different developmental and applied fields: mathematical knowledge is generally conceived of as objective and therefore ready made and absolutely valid knowledge« (Steinbring 2005, p. 14).

With the archetype of uniform, axiomatic mathematics, for the stoffdidaktic work the illusion was connected, tat also mathematics for teachers, students and pupils – thus school-mathematics – could ultimately be elaborated logically correct, consistently and for all teaching and learning processes definite and absolute. »The whole mathematical knowledge, ordered in this way, is in principle described with a single, universal language. This uniformity ... means approximately that the elementary concept of the number ›5‹ and the more abstract concept of the ›expectation of a binomially distributed random variable‹ are equal objects in the description by the mathematical set-language. This historical and timely shaped product of the mathematical knowledge corpus in its logical clarity, the construction from the simple to the complex and abstract, as well as by means of its uniform language, seems at the same time as the ideal preparation of the knowledge for its acquisition and its understanding – as it was for example also the maxim of the movement of so-called ›New Mathematics‹« (Steinbring 1998b, p. 161).

The presented stoffdidaktic work focused initially on the school mathematics of higher school grades (especially grammar school), and was then at the end of the sixties with didactical works in the frame of the movement of ›New Mathematics‹ (especially the works of Z. P. Dienes) extended to mathematics instruction in primary school. »The modernisation of mathematics instruction in primary school only started much later, about the year 1966, when the inventive ideas of Z. P. Dienes became familiar. ... We can speak of a modernisation of mathematics instruction in primary school and in grades 5 and 6 in three ways: First, it has already come to a reflection to more rigorous concepts already in primary school, which essentially rests on an analysis of the elementary-mathematical contents and their application situations. ... Secondly, it came to an input of new contents, which without doubt exceed what has been traditionally called calculating (Rechnen) . ... Thirdly, to a modernisation of mathematics instruction belongs essentially a reform of the picture of how to perform teaching, which has been evoked mainly by the introduction of the learning-oriented games in the instruction course.« (Griesel 1971, p. 8).

The stoffdidaktic analysis of school-mathematics of higher school grades as well as of primary school put uniform, logically consistent scientific mathematics (new contents and more rigorous concepts) into the centre of their work. Even if speaking of ›learning games‹, what is meant are mathematical activities on the basis of the subject matter structure and not predominantly child related activities among the children as trying, exploring and socially-discovering forms of learning.

For a characterising summary of the position of ›stoffdidaktik‹ described in this paragraph, the three questions (A, B & C) shall now be consulted and be answered in a general way. About the mathematical content, there clearly is the conception that ultimately a uniform, objective and then unchangeable content of teaching and learning according to the paradigm of scientific mathematics is to be elaborated in didactics. The teaching, learning and understanding processes of the participating persons (teacher and students) are orientated on the rigid subject matter structures, the teacher is the ›conveyor‹ of the didactically prepared content to the student(s), which are seen as passive receivers. The relations between the three elements of the didactical triangle are of an essentially linear nature: the mathematical knowledge arrives by means of the preparation and transfer from the teacher to the students. In the research paradigm of ›stoffdidaktik‹, the practised scientific elaboration of mathematical knowledge is the central and crucial means of steering and optimising the mathematical instruction, learning and understanding processes.

3. The synchronisation between the dynamics of knowledge development and the process of teaching and learning

The international criticism on new mathematics (Kline, Morris, 1973, Why Jonny can't add: The failure of new maths) lead also in Germany – in the social public as well as in mathematics education – to a longtime critical altercation with New Mathematics and especially with ›stoffdidaktik‹. Furthermore, the scientific debate about the status and the objects of a genuinely mathematics education science was led again and again and over a longer period (Steiner in ZDM 1974, Winter 1985, Wittmann 1992). Heinrich Winter, to cite only one prominent voice in this debate, states: »So-called *Sachanalysen* (›didactically oriented content analysis‹) can have a downright calamitous effect on the school reality, if these refer reductionistically solely to mathematics (perhaps even to assumed mathematics) and fade out other essential constituents of learning mathematics. ... [One] inevitably encounters problems of the goals and forms of learning itself, which are not or hardly explained in the Sachanalysen. ... In general: Sachanalysen are in danger of losing focus on the outer-mathematical reality and thus on the students' experience of the world, and this is only one pedagogical sin of such reductions« (Winter 1985, p. 80/81).

The relation between the mathematical learning content and the teaching and learning processes did not work in the way imagined from the perspective of ›stoffdidaktik‹. A new perspective onto the subject matter content needed to be developed, which took the sequential development and dynamics of teaching and learning *processes* into account. Hans Freudenthal has emphasized the process character of mathematics for learning in a paradigmatic way.

»It is true that words as mathematics, language, and art have a double meaning. In the case of art it is obvious. There is a finished art studied by the historian of art, and there is an art exercised by the artist. It seems to be less obvious that it is the same with language; in fact linguists stress it and call it a discovery of de Saussure's. Every mathematician knows at least unconsciously that besides ready-made mathematics there exists mathematics as an activity. But this fact is almost never stressed, and non-mathematicians are not at all aware of it« (Freudenthal 1973, p. 114).

Mathematics, as an activity, implies that learning becomes an active process in the construction of knowledge. »The opposite of ready-made mathematics is mathematics *in statu nascendi*. This is what Socrates taught. Today we urge that it be a real birth rather than a stylized one; the pupil himself should re-invent mathematics. … The learning process has to include phases of directed invention, that is, of invention not in the objective but in the subjective sense, seen from the perspective of the student« (Freudenthal 1973, p. 118).

Apart from focusing on a finished, generally valid mathematical (research) product, development processes are neither uniform nor universal or homogeneous. Subjective characteristics of those keeping the process going, as well as situated representations, notations and interpretations of mathematical knowledge, are manifold, divergent, and partly heterogeneous. In the process of developing mathematical knowledge, cultural contexts, subjective influences, and situated dependencies are both effective and inevitable, and are the reasons for an observable diversity and a nonuniformity of the emerging knowledge. In this regard, a learning student cannot be compared with a professional mathematician. The latter has many years of experience in mathematical communication with his colleagues, in the negotiation of the correctness of a mathematical assertion by using the communicative rules of a formal proof. Such professional communication aims directly at the uniform mathematical product in question, while the learning student is requested to develop and perfect such forms of mathematical communication with his classmates. The latter development process is essentially influenced by cultural aspects of teaching, by learning conditions which are subjective, by individual cognitive abilities, and by situated exemplary mathematical expressions and interpretations. Therefore, in the process of developing, learning and imparting mathematics, divergence and nonuniformity in understanding and interpreting are central.

The contrast between uniform scientific mathematics – oriented towards the generally valid (research) product – and the different perspectives and interpretations of mathematics produced in social environments for different application domains – tied up in situatedly framed development processes – becomes extremely apparent against the background of the different cultures in which mathematical knowledge is used and experienced. The culture of the researching and teaching mathematician, and the culture of mathematics teaching, face one another in an obviously distinct, and sometimes opposing, way. The role the Bourbakist mother structures play for the unity of mathematics cannot be understood by mere appropriation of the principles given by these structures. The culture of mathematical science and the historical development of mathematics form the necessary background for an understanding. These principles, of the unity of mathematical knowledge, cannot easily be transferred to school mathematics. With such an endeavor, school mathematics would lose their cultural background and become mere formalistic signs and formulas. In order to understand these signs and formulas, again, the formation of a new, distinct culture would be necessary, a kind of mathematical re-invention (Freudenthal). From the point of view, that mathematical signs, symbols, principles, and structures can only be meaningfully interpreted in the frame of a grown or newly emerging culture, one has to question the unity of mathematics in learning and teaching processes. If mathematical knowledge (signs, symbols, principles, structures etc.) can only be meaningfully interpreted in the frame of a specific cultural environment, then there is not simply one single, but many different forms of mathematics.

Wittmann discusses the problem of uniformity and difference of mathematics, not from the view of mathematical science like Bourbaki, but from the fundamental perspective of different scientific and practical fields, of society and culture. Wittmann distinguishes between specialized, scientific mathematics and the general social ›phenomenon‹ mathematics.

»[One] ... must conceive of ›mathematics‹ as a broad societal phenomenon whose diversity of uses and modes of expression is only a part reflected by specialized mathematics as typically found in university departments of mathematics. I suggest a use of capital letters to describe MATHEMATICS as mathematical work in the broadest sense; this includes mathematics developed and used in science, engineering, economics, computer science, statistics, industry, commerce, craft, art, daily life, and so forth according to the customs and requirements specific to these contexts. Specialized mathematics is certainly an essential element of MATHEMATICS, and the broader interpretation cannot prosper without the work done by these specialists. However, the converse is equally true: Specialized mathematics owes a great deal of its ideas and dynamics to broader scientific and societal sources. By no means can it claim a monopoly for ›mathematics‹.

It should go without saying that MATHEMATICS, not specialized mathematics, forms the appropriate field of reference for mathematics education. In particular, the design of teaching units, coherent sets of teaching units and curricula has to be rooted in MATHEMATICS.

As a consequence, mathematics educators need a lively interaction with MATHEMATICS, and they must devote an essential part of their professional life to stimulating, observing, and analyzing genuine MATHEMATICAL activities of children, students and student teachers. Organizing and observing the fascinating encounter of human beings with MATHEMATICS is the very heart of didactic expertise and forms a natural context for professional exchange with teachers.

As a part of MATHEMATICS, specialized mathematics must be taken seriously by mathematics educators as one point of view that, however, has to be balanced with other points of view« (Wittmann 1995, p. 358/9).

On the basis of this position about the role of mathematical knowledge in instruction processes, Wittmann characterises didactics of mathematics as a ›Design Science‹ (1998, 2001): »The clear structural delineation of mathematics education as a *design* science from the related sciences underlines its specific character and its relative independence. Mathematics education is not an appendix to mathematics, nor to psychology, nor to pedagogy for he same reason that any other design science is not an appendix to any of its related disciplines. Attempts to organize mathematics education by using related disciplines as models miss the point *because they overlook the overriding importance of creative design for conceptual and practical innovations.*« (Wittmann 1998, p. 94). In German-speaking mathematics education – especially concerning teacher education at universities and teachers' further education – Wittmann is a protagonist for a new perception on the role and the meaning of mathematical knowledge for teaching and learning processes, which critically distances itself from new mathematics.

At the »Institute for Didactics of Mathematics (IDM)«, founded at the University of Bielefeld in 1973, fundamental researches about mathematics education positions, problems and research questions were carried out in three working groups of scientists. In the working group »Mathematics Teacher Education« (Michael Otte and members of the research group, 1981), mainly two central research approaches in mathematics education were pursued: (1) the particular epistemological nature of mathematical knowledge and (2) the central role of the teacher within mathematical teaching and learning processes.

Historical, philosophical and epistemological analyses have been elaborated as a basis for characterizing mathematical knowledge ultimately as *theoretical* knowledge. A central criterion of theoretical mathematical knowledge – also observable in the course of its historical development – lies in the transition from pure object, or substance thinking, to relation or function thinking.

»Every one knows that there are things, and relations between things. Classical mathematics is primarily and essentially a *thing*-mathematics. The Western is primarily and essentially a *relation*-mathematics, a mathematics of functions – that is, relations –, in accord with the Faustian saying of Henri Poincaré, that science can not know ›things‹ but only ›relations‹« (Keyser 1932/33, p. 193).

The transition from a *substance concept* to a *relational concept* is a central part of Ernst Cassirer's epistemological philosophy.

»… the theoretical concept in the strict sense of the word does not content itself with surveying the world of objects and simply reflecting its order. Here the comprehension, the ›synopsis‹ of the manifold is not simply imposed upon thought by objects, but must be created by independent activities of thought, in accordance with its own norms and criteria« (Cassirer 1957, p. 284).

And in another passage, Cassirer writes: »It is evident anew that the characteristic feature of the concept is not the ›universality‹ of a presentation, but the universal validity of a principle of serial order. We do not isolate any abstract part whatever from the manifold before us, but we create for its members a definite relation by thinking of them as bound together by an inclusive law« (Cassirer 1923, p. 20).

This understanding of theoretical mathematical concepts as referring to relations, and not to objects or to the empirical properties of objects, constitutes the basic step towards developing mathematics education into a scientific discipline.

»For didactics, for instance, it is obvious that the didactic problem in its deeper sense, that is in the sense that it is necessary to work on it scientifically, is constituted by the very fact that concepts will reflect relationships, and not things. Analogously, we may state for the problem of the application of science that it will become a real problem only where the relationship between concept and application is no longer quasi self-evident, but where to establish such a relationship requires independent effort« (Jahnke & Otte 1981, pp. 77/78).

A perception that mathematical knowledge does not reflect things, but relations, implies a differentiated view onto teaching and learning mathematics as independent activities of the participating persons. Thus, the role of the teacher comes to the fore. »A description of the requirements on the teacher and the teaching activity has been attempted in the debate about the relation between *teaching* and *learning*. From this debate, one can record as a consequence that ›teaching‹ cannot be derived from the descriptions of ›learning‹ - and that according to the opinion of many authors the developmental status of learning theories is more advanced than the one of teaching theories. After all, the conception that the contents of teacher education should essentially consist of insights about the student's learning process is by all means common.

What is the specific of teaching? The specific of teaching lies within the content of the activity, which aims at effectuating learning. Every theory of academically institutionalised education thus presupposes a concept of teaching and cognition, but furthermore requires also perceptions about the questions, by which mechanisms the teaching/learning process leads or shall lead to an interactionally imparted forming of the learner.« (AG Mathematiklehrerbildung 1981, p. 57).

According to this perspective, theoretical and empirical works about the particularity of the teacher activity have been carried out in the mentioned working group of scientists of the IDM Bielefeld (see for instance the selection: Bromme 1981, 1992, Bromme & Seeger 1979). These concepts and works about the activity of the mathematics teacher reveal in particular that within the didactical triangle, the teacher and his role are determinated neither by the mathematical knowledge nor by the learning students. For instance, Bromme (1981, 1992) analyses central aspects of the teacher activity (for example the preparation of mathematics instruction) under the perspective that teachers are to be regarded as experts of their professional field of work. In addition to the two essential fields of professional teacher knowledge, the ›content knowledge‹ and the ›pedagogical content knowledge‹ (according to Shulman 1986), Steinbring (1998) elaborates the particularity of ›epistemological knowledge for mathematics teachers‹ with a view to the *theoretical* and *dynamic* character of mathematics. This knowledge concerns insights about the particular epistemological nature of mathematical knowledge for teaching and learning processes, which are not contained in the ›pedagogical content knowledge‹, which Shulman briefly describes as follows: »… the ways of representing and formulating the subject that make it comprehensible to others« (Shulman 1986, p. 9).

Furthermore, during this time, the meaning and independent role of the learning child with its cognitive predispositions moved more and more to the centre of didactical research – a position, which had already been taken for a longer time in primary school didactics. The didactical research and positions about the role of the student and learning individual have a long (also international) tradition. The discussion about the development of mathematics education in Germany shall at this point only be characterised briefly by the following general quote. In a summarizing main lecture on the Federal Congress for ›Didaktik der Mathematik in Osnabrück‹ in 1991, Peter Sorger sums up the state of the (German) mathematics education works about the learning child: »Today, we know so much more especially about the individual primary school child, about its cognitive activities, about its thinking, about the initiation and course of mathematical learning processes, about the influences of the individual learning history onto new learning situations, about the variety of possible thinking and solution strategies, in which the adults' perceptions are always in danger of cutting too short. The diagnose, analysis and therapy of learning difficulties are also thoroughly researched« (Sorger 1991, p. 39). The researches about this topic in particular use methods from reference disciplines, and they are not reducible to mathematical works, i.e. they essentially contribute to an independent research profile of mathematics education.

Again, the three questions (A, B & C) shall be asked and answered in a general way, in order to characteristically sum up the positions about mathematical instruction (respectively teaching and learning processes) described in this paragraph, in a critical distance from ›stoffdidaktik‹. The mathematical content is interpreted more manifold and especially its dynamic and procedural character is emphasised. (School-)Mathematical knowledge is not identical with the scientific mathematical research knowledge, but it is at the same time *theoretical* knowledge, it is subject to a particular epistemology (also in the frame of the activities of teaching and learning). Only this developmental aspect of mathematical knowledge makes it possible to construct a kind of ›compatibility‹ with the developing learning activities by the students respectively with the teacher's teaching activities. The more differentiated mathematics education perceptions negate an immediate dependence of the teacher from the mathematical knowledge and of the student from the instructing teacher. Learning mathematics is in a certain way autonomous, a »socially and actively discovering, independent learning by the students« (Wittmann, and also Freudenthal). And teaching is analysed as an independent activity (AG Mathematiklehrerbildung of the IDM Bielefeld).

The three elements of the didactical triangle, (1) the mathematical knowledge, (2) the student and (3) the teacher stand ›apart‹ and gain independence as well as their own dynamics with new didactical research questions. The relations between these elements are of a rather indirect nature. For instance, the teacher is now regarded rather like a moderator or initiator of learning processes, and the student is conceded his own responsibility for his mathematical understanding and learning processes. The developing mathematical knowledge becomes manifest in different ways in different using and teaching/learning practices – it is no longer consistently and universally given, for example on the basis of the Bourbakian structure types.

The ›steering‹ of the students' learning processes by the teacher can no longer be perceived as a mechanical conducting. The ›functioning‹ of the didactical triangle now rather represents a reciprocal process between its three elements and not a linear or circular movement of mathematics via the teacher to the students etc. The mathematical knowledge – now in its new interpretation as *theoretical* knowledge within a *development process* – remains important, but shows itself in different characteristics in learning and teaching activities; but also the student's learning activities and the teacher's teaching activities have an essential influence onto the whole process.

At first, didactical researches concentrate rather on the three – each relatively autonomous – elements of the didactical triangle, (1) the mathematical knowledge, (2) the students and (3) the teacher; only with the beginning of mathematics education interaction research the co-action of the three elements is explicitly taken serious as the central didactical research object.

4. Mathematics education research and mathematical teaching-learning-practice as independent institutional systems

For a long time, mathematics education research explicitly or implicitly took the standpoint that the mathematics teaching practice in school had to follow strictly the input and insights constructed by mathematics education. This standpoint was represented explicitly in the so-called ›stoffdidaktik‹; also in the frame of didactical works, which emphasise the procedural character of mathematical knowledge and of mathematical teaching and learning situations, there were and are perceptions according to which instruction practice could be guided and improved essentially by means of the insights of educational research.

Mathematics education, like any content matter pedagogy, is faced with the tension between scientific research and constructive development work. This problematique has been discussed intensively for a long time, for instance in the scientific debates about the so-called ›Theory-practice-problem‹ (Bazzini 1994; Even & Loewenberg Ball 2003; Steinbring 1994; Seeger & Steinbring 1992; Verstappen 1988). There exists the general claim that educational research and development works can always also bring about support, positive influences and direct improvements for teaching and learning processes in school (and university).

Facing this complementary task of research and constructive development, mathematics education is confronted with the fundamental question: »What is the *particular nature* of the relation between theory and practice?«. A longtime, traditional perception, according to which knowledge and contents of subject matter are thoroughly researched and elaborated in content related educational theory, in order to be then transferred into school practice, has been decidedly criticised and replaced by other conceptions.

An essential criticism – on this position, which is intimately connected with ›stoffdidaktik‹ –developed by means of the works of the researchers group around Heinrich Bauersfeld at the IDM (Bielefeld). Since around the beginning of the eighties, such researches, in which everyday mathematics teaching as autonomous social events were taken serious and analysed under an interactionistic perspective, were more and more performed in mathematics education (e.g. Bauersfeld, 1978, 1988; Cobb & Bauersfeld 1998; Krummheuer 1984, 1988; Maier & Voigt, 1991, 1994; Voigt, 1984, 1994). Mathematics instruction, taking place every day, is seen as an independent culture, which is neither completely nor directly determined by the scientific discipline ›mathematics‹, nor can be directly guided and improved by means of the mathematics education insights.

Jörg Voigt (1996) calls this the ›turn to everyday life‹ of the authentic classroom in mathematics education: »… the ›*turn to everyday life*‹ … with its criticism on ›holiday didactics‹ … contained the claim of assigning a greater meaning than before to the features of everyday instruction. In ethnographic instruction observations and interpretative studies, one saw a corrective for instruction conceptions, which emerge at the didactical desk; one was disillusioned by the effects of the school reforms (see among others the ›New Mathematics‹) and wanted to understand better the surprising stability of everyday instruction, its own progress and its traditions. At the same time, there was the hope of being able to better connect with the experience and the problem awareness of the practitioners with softer methods of empirical research« (Voigt 1996, p. 384).

The works of the group ›interpretative instruction research‹ (around H. Bauersfeld) were exposed to hostility at the beginning of the eighties – especially from the people of ›stoffdidaktik‹ – as they increasingly pointed out that it was an illusion that mathematics education with its insights and constructive materials could directly affect and thus improve instruction. Mathematics instruction within the institutional context of school and socially organised knowledge transfer is subject to its own laws, requirements and aims, which cannot be directly changed from the outside.

(School) Practice and (content related educational) science need to be seen as two relatively autonomous institutions and fields of work, between which there are no direct possibilities of influence or change (see about this Bartolini-Bussi & Bazzini 2003; Krainer 2003; Scherer & Steinbring 2003; Steinbring 1994; 1998). Each of the two fields is subject to its own expectations and aims, as well as to system-internal requirements and norms, which cannot be invalidated from the outside in order to apparently be able to directly interfere into and to purposefully regulate within the other field.

The relative separation and autonomy of (content related educational) theory and (school) practice, however, does not mean that there are no reciprocal actions between the two at all. Rather, in the relation between theory and practice, the respective other field can be seen as a necessary environment, in which irritations and stimulations occur, which indirectly animate the one field in order to implement changes, alternative ways of proceeding and further developments. What is important here is to notice that such changes within (school) practice – but also within content related educational theory – must ultimately occur and establish themselves from the inside and ›out of themselves‹, and in order for this to happen, irritations and stimulations from the outside are helpful and necessary, yet they are no deterministic influencing instruments.

Under this fundamentally changed perspective onto the ›theory-practice-problem‹, the didactical triangle obtains a different orientation function for mathematics education research. It does no longer represent an ideal paradigmatic schema at which everyday instruction must be measured, but it becomes an instrument for the analysis of really existing mathematics instruction, in which the reciprocal interconnectedness between the three relevant elements participating in the instruction process are systematically captured.

In the works about interpretative classroom research, the social interactions and their patterns and mechanisms stood in the centre of the research interest, the mathematical teaching and learning content was in principle faded out. Thus, in particular the relation between the two elements of the didactical triangle (2) the student and (3) the teacher within the frame of everyday instruction events was thematised.

As a result of the criticism of ›stoffdidaktik‹, the interactionist perspective relies mainly on two (until then neglected) basic aspects: the *learning child* (in the classroom) and the *interaction between the learner and the teacher*. In this research context, one has to distinguish between two theoretical perspectives: »The one is an individual-psychological perspective which emphasizes the learner's autonomy and his cognitive development and which leads to the concept of student-oriented, ›constructivistic‹ mathematics instruction. The other is a collectivistic perspective which criticizes the ›child-centered ideology‹ of the first perspective and understands learning mathematics as the socialization of the child into a given classroom culture ...« (Voigt 1994, p.78).

These two research perspectives are thus based on reference to different scientific disciplines. The individual-psychological perspective relies, for example, on cognitive psychology as well as on radical constructivism (von Glasersfeld 1991); and the collectivistic perspective uses sociological and ethnographic theories. In the analyses of mathematical interactions, one or the other of these two theoretical orientation is often emphasized. (Concerning the individual-psychological perspective see e.g. Cobb, Yackel & Wood (1991); and for the collectivistic perspective see Solomon (1989)). An overemphasis of either the individual-psychological or the collectivistic perspective was a major critique and a starting point for the working group around H. Bauersfeld to develop a theoretical concept explicitly bringing together the individual, cognitive perspective and the collective, social perspective as a basis for qualitative analyses of interaction.

On the one hand, it is stated that a single student cannot discover all school knowledge by himself. »Culture, we can say, is not discovered; it is traded or falls into oblivion. All this indicates for me that we should rather be more careful when talking about the discovery method or about the conception that discovery is the basic vehicle of instruction and education« (Bruner 1972, p. 85). On the other hand, it is considered doubtful that effective participation in social interaction patterns can lead to successful mathematics learning.

»In everyday lessons interaction patterns often can be reconstructed in which the teachers influence every step of the students' activities without creating favorable conditions for the student to make desirable learning processes in problem solving and developing concepts.... We should resist the temptation of identifying learning mathematics with the student's successful participation in interaction patterns (Voigt 1994, p.82).

Consequently, an interaction theory has been developed, in which both perspectives were connected to each other. »[A]n interaction theory of teaching and learning mathematics [offers] a possibility of regarding social aspects of learning mathematics and at the same time of avoiding the danger of overdoing the cultural and social dimensions. For the interaction theory emphasizes the processes of sense making of individuals which interactively constitute mathematical meanings. The interaction theory of teaching and learning mathematics uses findings and methods of micro sociology, particularly of symbolic interactionism and ethnomethodology. … Of course the interaction-theoretical point of view does not suffice if one wants to understand classroom processes holistically« (Voigt 1994, p.83).

The interpretation that mathematical knowledge as a *theoretical* – and not empirically fixed – knowledge *develops* and changes in such development processes regarding its epistemological status, by means of becoming more abstract, general and universal, makes it possible in social processes of teaching and learning, in comparison to the historical development of mathematics, which was bound into social and cultural contexts, to understand the interactive generation of mathematical knowledge as a relatively independent procedure within the frame of the instruction culture.

The interaction-research approach of the *social epistemology of mathematical knowledge* (Steinbring 2005) understands itself as an important, independent complete model inasmuch as the particularity of the social existence of mathematical knowledge is an essential component of this theoretical approach of interaction analysis. In this theoretical conception of the social epistemology of mathematical knowledge, the epistemological particularity of the subject matter ›mathematical knowledge‹ dealt with in the interaction constitutes a basis for its theoretical examination. In this theoretical investigation mathematical knowledge is seen from a different perspective: the subject matter of ›mathematics‹ is, according to the considerations in the previous considerations, not understood as a pregiven, finished product, but interpreted according to the epistemological conditions of its dynamic, interactive development.

Every qualitative analysis of mathematical communication always has to start – explicitly or implicitly – from assumptions about the status of mathematical knowledge. There are different ways of coping with this requirement. Epistemology-based interaction research in mathematics education proceeds on the assumption that a specific social epistemology of mathematical knowledge is constituted in classroom interaction and this assumption influences the possibilities and the manner of how to analyze and interpret mathematical communication. This assumption includes the following view of mathematics: Mathematical knowledge is not conceived as a ready made product, characterized by correct notations, clear cut definitions and proven theorems. If mathematical knowledge in learning processes could be reduced to this description, the interpretation of mathematical communication would become a direct and simple concern. When observing and analyzing mathematical interaction one would only have to diagnose whether a participant in the discussion has used the ›correct‹ mathematical word, whether he or she has applied a learned rule in the appropriate way, and then has gained the correct result of calculation, etc.

Mathematical concepts are *constructed* in interaction processes as symbolic relational structures and are coded by means of *signs and symbols*, that can be combined logically in mathematical operations. This interpretation of mathematical knowledge as »symbolic relational structures that can be consistently combined« represents an assumption which does not require a fixed, pre-given description for the mathematical knowledge (the symbolic relations have to be actively constructed and controlled by the subject in interactions). Further, certain epistemological characteristics of this knowledge are required and explicitly used in the analysis process; i.e. mathematical knowledge is characterized in a consistent way as a structure of relations between (new) symbols and reference contexts.

The intended construction of meaning for the unfamiliar, new mathematical signs, by trying to build up reasonable relations between signs and possible contexts of reference and of interpretation, is a fundamental feature of an epistemological perspective on mathematical classroom interaction. This intended process of constructing meaning for mathematical signs is an essential element of every mathematical activity whether this construction process is performed by the mathematician in a very advanced research problem, or whether it is undertaken by a young child when trying to understand elementary arithmetical symbols with the help of the position table. The focus on this construction process allows for viewing mathematics teaching and learning at different school levels as an authentic mathematical endeavor.

In the frame of the epistemologically-oriented qualitative analysis of mathematical classroom interactions, the subject of teaching and learning – mathematical knowledge – is taken into account as an important element within the didactical triangle. For empirical, interpretative research, the didactical triangle takes a descriptive function – and it has no prescriptive function – with which guidelines for instruction practice are provided. As a descriptive schema, the didactical triangle serves for characterising an essential and complex – not further dissectible – fundamental object of mathematics education research, namely (everyday) mathematical interactions and communications within teaching and learning processes.

To sum up, one can ascertain the following alongside the three questions (A, B & C). The three elements of the didactical triangle, (1) the mathematical knowledge, (2) the student and (3) the teacher, are seen in the institutional context of the joint interaction as relatively independent ›systems‹, which are engaged in reciprocal actions with each other. The mathematical interactions between teacher and students take place between autonomous subjects, who are aware of each other during the reciprocal communication, but who cannot directly influence the psyche or the consciousness of the other. The communicated and negotiated mathematical knowledge is interactively constructed within this social context – on the basis of its epistemological basic conditions of consistence and structure.

Accordingly, the teacher continues to take the role of a moderator or a facilitator of learning occasions for the students, and the student continues to be responsible for his own understanding process and participates by means of social and actively discovering mathematics learning. The instructional communication process emerges and constitutes itself within the actual execution of the teaching and learning process, it cannot be planned and prepared en detail beforehand. The teaching-learning-object mathematics develops within the social interaction, and it is in different ways the ›subjective property‹ of the persons taking part in the interaction.

The question about the decisive means for positively changing and affecting the teaching, learning and understanding processes (C) obtains a more differentiated background. Changes and improvements cannot take place from the outside or by means of a direct intervention. Changes can only be encouraged in the participating autonomous systems and then need to be continued and realised within the systems themselves. This concerns the learning student, to whom the teacher can ultimately only offer learning opportunities to learn himself. But this is also true for the teacher and the development of his professional teaching activity in connection with mathematics education research.

5. Mathematical knowledge in Teaching – A case for illustrating the epistemology-based interaction view on teaching learning processes

In the following, the case of short teaching episode (see Appendix), is used to illustrate examplary how, from the perspective of epistemologically-oriented empirical instruction research, the three elements of the didactical triangle, (1) the mathematical knowledge, (2) the student and (3) the teacher, autonomously and interactively generate mathematical knowledge within this social situation. A summarizing epistemological analysis shall clarify if and how the theoretical character of mathematical knowledge is accommodated in the interaction (the following account is based on Nührenbörger & Steinbring 2006).

From the very beginning, mathematics teaching and mathematics learning deals with the use and interpretation of mathematical signs, symbols and symbol systems. Using the example of the number concept and its epistemological characteristics, the role of mathematical signs/symbols shall be characterised as far as it becomes important for the following epistemological analysis of mathematical interactions. Initially, one can distinguish two essential functions for mathematical signs/symbols:

»(1) A semiotic function: the role of mathematical signs as ›something which stands for something else‹. (2) An epistemological function: the role of the mathematical sign in the context of the epistemological interpretation of mathematical knowledge.« (Steinbring 2005, p. 21)

A comparison between linguistic and mathematical signs reveals the following concerning the first function. The linguistic sign or word ›school‹ first stands for a concrete school – maybe the school, which the students attend. But with ›school‹, one can also designate a big number of different concrete schools – of the same or of a different type. This relation between the word ›school‹ and many concrete schools also covers the ideal construct of the general concept ›school‹ as a place of institutionalised teaching and learning scientific knowledge – and a concrete school is the realisation of this abstract idea. Furthermore the sign ›school‹ can be written in different forms (cursive, block letters, etc.) or languages (école, Schule, scuola, etc.) without there being a change in the illustrated relation between the linguistic sign and the concrete referents or in the abstract idea.

The mathematical sign ›4‹ stands for the conceptual number ›4‹, and that ultimately is an abstract conceptual idea from the beginning. In order to facilitate and to activate child-accordant mathematical learning and understanding processes, there is a multitude of didactical situations and materials to which the sign ›4‹ could relate.

One example for such a referential relation between the sign ›4‹ and an object, which this signs stands for, could be the use of little coloured chips: 

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Insofar, the sign ›4‹ relates to the four chips, but does not designate these as the actual objects (as for example the word ›school‹ designates the concrete school of a student), but ultimately ›stands for something else‹ which is meant by the four coloured chips, namely always the abstract concept of the number ›4‹. Comparable to the different writings of the word ›school‹, the mathematical sign ›4‹ can be written in other ways and languages: ›4‹, › IV ‹, ›100‹ (in the binary system), etc. or ›vier‹, ›quatre‹, ›quattro‹, etc. on the one hand, this difference to linguistic signs – namely that mathematical signs/symbols ultimately always relate to a universal mathematical conceptual idea and not to ›concrete mathematical numbers‹ (for example different materials) – illustrates the special epistemological character of mathematical signs.

The mediation between signs and structured reference contexts requires a conceptual mediation (Steinbring 2005, p. 22). The conceptual idea of ›natural number‹ is needed for regulating the relation between the signs/symbols and their accompanying reference contexts. What is this conceptual idea of the number concept?

In contrast to an empirical understanding of numbers as representing concrete objects or as names of sets, such a conception is fundamentally criticized from philosophical and epistemological perspectives. Paul Benacerraf (1984) for instance states that numbers can neither be objects nor names for objects. »I therefore argue, … that numbers could not be objects at all; for there is no reason to identify any individual number with any one particular object than with any other (not already known to be a number)« (Benacerraf 1984, pp. 290/1). But if numbers are not objects, what else are they? »To be the number 3 is no more and no less than to be preceded by 2, 1, and possibly 0, and to be followed by 4, 5…… Any object can play the role of 3; that is any object can be the third element in some progression. What is peculiar to 3 is that it defines that role - not being a paradigm of any object which plays it, but by representing the relation that any third member of a progression bears to the rest of the progression« (Benacerraf 1984, p. 291).

In the following, the development of a mathematical interaction and collaboration between two young students will be analysed. Initially, the two boys are working together, then, the teacher joins them for a short phase and intervenes in the mathematical interaction. This first part presents an epistemological analysis for comparing the autonomous mathematical interpretations of the two boys with the mathematical interpretations, which are discussed later in the teacher's intervention.

At the beginning of the lesson, the ›number house 7‹ (in the roof) is investigated with the whole class. The task is to find all possible addition tasks for the number ›7‹, and then to write them down in ›floors‹. The children then are working in pairs of two. On their desks, there are two roofs with the numbers ›8‹ and ›14‹ and a number of paper strips as floors. The two children are to find as many floors as possible. They are building the number house in their own chosen sequence and if necessary use chips.

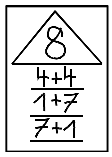

Fig. 2: Number House ›8‹

KLAUS and sönke are working together at the number house ›8‹ (Fig. 1). They have already found the decompositions ›4 + 4‹ (written down by KLAUS), ›1 + 7‹ (written down by sönke) and ›7 + 1‹ (written down by KLAUS). In the following, the mathematical interactions of these two boys will be presented in a shortened way (see the transcript in the appendix).

Phase 1.1 [1-12]: *KLAUS explains the mathematical writing of the number Six*

KLAUS and sönke are looking for further additive decompositions to the number ›8‹ in the roof of the number house. Now sönke is to write down the next decomposition.

KLAUS asks sönke to write down: six plus two. sönke agrees and writes down on a paper strip: d+2. KLAUS asks which ›letter‹ that (d) is supposed to be; sönke does not understand and KLAUS asks which number it is supposed to be. sönke says: Six. KLAUS adds that the sign (d) looks a little funny; he ›writes‹ down a mathematical 6 on the table with his index finger and remarks that sönke's sign looks like a ›d‹. The according paper strip is taken away and sönke writes down on a new empty paper strip (correctly) 6 + 2.

It is KLAUS's turn and he writes down the decomposition 2 + 6 on a paper strip.

Phase 1.2 [13 – 36]: sönke finds an exchange task to 4+4.

KLAUS considers which other decomposition is possible and suggests to sönke to write down: five plus three. At the same time, he remarks that there is nothing else left. sönke writes down 5 + 3 on an empty paper strip. While doing this, sönke comments that (in the alternating sequence) he should ›be at the end‹, and then they are finished. Everybody, KLAUS and sönke, has then written down the same amount of decompositions.

KLAUS takes the pencil and writes down the decomposition 3 + 5 on an empty paper strip. KLAUS asks whether there is anything else (further decompositions) and he declares that there is nothing else. Meanwhile, sönke is attentively regarding all labeled paper strips and moves them up a little one after the other.

sönke opens this sub-phase with the question: »Do you know what I'm simply going to write down there?« (26) and then says again that he has to do once more and then they are finished (with this number house). He then tells his suggestion: »four plus four« (30). KLAUS surprised asks »Again?« (31) and sönke explains that they have every number again, just the other way round. Then, ›four plus four‹ would also be the other way round, one just cannot see it. Following this, he changes the outer writing form of the already noted 4+4 into 4+4 and writes it on an empty paper strip like this. KLAUS is surprised and does not contradict; sönke says that they now have everything.

Phase 2.1 [71 – 92]: *The teacher pointes to the duplication of the task* 4 + 4 *and missing decompositions*

In the following scene, the two boys are working on the number house with the roof number 14. They have already found a number of decompositions of the 14 and written them down on paper strips. When the teacher comes to their table, sönke writes down the decomposition 10 + 4. This is when the episode begins. The students are elaborating the decomposition ›10 + 4‹. In this moment, they realise that the teacher is approaching and cast him a look. KLAUS remarks that that have finished their work on the first house. The teacher agrees and waits until the students have finished their current exercise. Meanwhile, he thoroughly examines the paper strips of the first house and moves them apart a little. KLAUS ›dictates‹ sönke the decomposition step by step and accordingly, sönke notes this on a paper strip. They want to switch with writing the next decomposition. Here the teacher intervenes.

The teacher stops the further work on the new number house. He asks the students to check the decompositions in the old number house, and to see whether double tasks appear; these should be taken out. KLAUS points at the upper strip with the decomposition 4+4 and then at the lower one with the decomposition 4+4: »These two«. The teacher confirms this.

KLAUS explains that they did it in the same way as with the two decompositions 6 + 2 and 2 + 6 and asks whether one was not allowed to do this with 4 + 4. The teacher assumes that KLAUS thought ›4+4‹ and ›4+4‹ to be ›exchange addition tasks‹. KLAUS blames it on sönke: »sönke first thought so (*points at S*).« (86)

The teacher points at the two paper strips with the decompositions 6 + 2 and 2 + 6 and explains that these are the same numbers, but in a different sequence. Thus, as the teacher explains, these are different tasks. Then he alternately points at the paper strip with 4+4 and the paper strip with 4+4 and says in comparison to the two previous decompositions that one could not distinguish these two, right? KLAUS agrees and says that he will take them away, what he does immediately.

In the conversation with KLAUS, the teacher confirms this. At the same time, sönke remarks quietly and by himself – neither the teacher nor KLAUS realise this »But these are different fours.« (91). The teacher closes this phase remarking that in the number house ›8‹, there are still some (decompositions) missing, which should be found,

A summarizing analysis of the episode

Initially, KLAUS and sönke reach an understanding about how the number six is usually written as a mathematical sign. KLAUS explains that the writing d is not correct and shows the common form: 6. The referential explanations for the correct mathematical sign for ›six‹ take place on a *conventional* level – no *genuinely mathematical* relations for this explanation are used.

Then, sönke constructs a second decomposition of 8 into 4 + 4. All other decompositions appear twice; this second decomposition is different by means of a new writing: first 4+4 and 4+4. This explanation attempts to use mathematical relations. On the one hand, analogies to the other decompositions of the 8 are constructed. The same ›structure‹ for the number of decompositions is requested. In the changed writing of the four, rather *conventional* conditions are brought into play.

Later, the teacher intervenes and discusses reasons why the decomposition 4 + 4 is not allowed to not appear twice: the duplicated decomposition into 4 + 4 is a *double task*, but other decompositions (6 + 2 & 2 + 6) can be seen as two *different* tasks. KLAUS accepts immediately. This explanation is essentially based on the mathematical term ›*(addition) task*‹ and on the designations ›*different* and *double* or *equal tasks*‹, which are thus superfluous. Thus this explanation contains specific mathematical aspects in an elementary form.

KLAUS points to an analogy with other decomposition (6 + 2 & 2 + 6), in order to write down a second decomposition into 4 + 4. This justification, again, is ›close to mathematics‹, for example a consistency with the other decompositions is required.

The teacher underlines that his justification that 4 + 4 may only appear once: The term ›exchange task‹ makes sense only for other decompositions, not for 4 + 4. Furthermore: with exchange tasks, there are the same numbers, but in a different sequence, and thus these tasks are ›different‹. These explanations use elementary mathematical concepts: (addition) task, exchange task, sequence of the numbers (summands) within the tasks, etc. Of course, there are conventional parts as well, but it is important that with these elementary concepts ›*mathematical* relations‹ between the numbers (in operative connections) are meant or should be meant.

Ultimately sönke insists on his explanation that his different writings of ›4‹ mean that the fours are different and also the two decompositions, a justification, which is founded *purely conventionally* and not specifically *mathematically*.

The two students KLAUS and sönke have – ultimately in an equal form – interpreted numbers and arithmetical operations. KLAUS has explained to sönke how the mathematical signs of the number six is written. Later, sönke has explained and carried out his suggestion to write down a second – differently written – decomposition of the 8 into 4 + 4; KLAUS has accepted this (cf. Dekker, Elshout-Mohr & Wood 2006). The two students have in this way constructed personal and cooperative mathematical knowledge about numbers. With the writings of numbers, like d (for six) as well as the decompositions 4+4 and 4+4, a fundamental epistemological problem is connected under the outer typographical surface: Mathematical signs/symbols – also with completely different notations – do not relate to ›different concrete objects‹ (learning materials e.g.), but are necessary for the coding of the abstract mathematical concept or the theoretical mathematical knowledge. How can this demanding topic be made understandable for young students at the beginning of elementary school?

In the second interaction phase, the teacher intervenes. Very quickly, a basically changed interaction behaviour between the two students can be observed. The teacher communicates almost exclusively with the older student, KLAUS. The younger one, sönke, can hardly interfere in the discussion. In the common work of the two boys, an equal, collaborative communication and work could be observed initially, which now, with the teacher's intervention, switches to a hierarchic, non-equal communication, strictly focused on KLAUS. The teacher does not gather information about the discussions and knowledge constructions of the children. He essentially checks the progress and the ›correctness‹ of the previous work. The ›double tasks‹ (the wrong ones) shall be found and removed; additionally, further tasks to the number 8 are to be found before continuing the work on the next number.

The mathematical explanations given by the teacher in order to make understandable why there could not be a second decomposition for 4 + 4, are essentially the following terms: tasks, exchange task, sequence of the numbers (summands) within the tasks. KLAUS accepts these explanations quite spontaneously and takes away the second strip with ›4+4‹; sönke however insists – rather quietly – that there are different numbers and decompositions. Ultimately, the fundamental epistemological problem is not really solved by means of the teacher's intervention. KLAUS accepts that in exchange tasks, same numbers must appear in a different sequence (teacher), that thus in ›4+4‹ there is no different sequence. He seems to accept the teacher's explanation mainly because of his authority; one cannot discover whether he has gained a far-reaching understanding of the difficult mathematical interpretation problem. For sönke, however, the fours remain different.

The difficult epistemological problem whether there is a second decomposition for 4 + 4 also remains open in the interaction with the teacher and is not really solved. With an empirical interpretation and a visual understanding of the first natural numbers (1, 2, 3, 4, 5, 6, 7, …), that are connected with concrete working material – like the chips used in the lesson here – the children can interpret these numbers as signs to count, put together and separate many concrete numbers. Within this frame, it can be justified that they add ›first the second four chips‹ and ›then the first four chips‹ in the act of calculating and thus reach a different, distinguishable decomposition of 4 + 4. But then, numbers are directly bound to empirical objects. The act of calculating carried out on concrete objects has to be distinguished from the abstract arithmetical operation of addition (cf. Dörfler 2004): in abstract mathematics it is always about the same abstract operation 4 + 4, even if the notation of the signs should be different.

With a view on the didactical triangle as a descriptive instrument in order to label the essential elements and their reciprocal actions within mathematical teaching and learning processes, the new interpretation from a mathematics education research perspective becomes clear in this episode: Mathematical knowledge is interactively constructed by the participants on the basis of specific epistemological conditions, which are in effect also within instructional learning processes, and which there lead to a socially developed epistemology of (school) mathematical knowledge.

(Remark: Many paragraphs of this contribution – in particular chapters 1 through 4 – are based on Steinbring 2005)

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Appendix

Transcript scenes of a mathematical classroom interaction

KLAUS and sönke find decompositions for the number houses ›8‹ and ›14‹

The topic of the lesson **»Find all decompositions of a given number«.** In the introduction, a number house (›7‹) having been discussed during the last lesson was taken as the starting point for the problem that as many addition tasks leading to the number in the roof as possible shall be found. On the children's desks, there are two roofs with the numbers ›8‹ and ›14‹ as well as a multitude of paper stripes meant as floors. The children are building the number houses in their own chosen sequence and use coloured chips as required.

KLAUS and sönke are working together at the number house ›8‹. They have already found the dissections 4 + 4 (written down by KLAUS), 1 + 7 (written down by sönke) and 7 + 1 (written down by KLAUS).

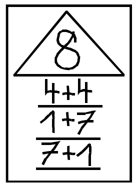

Fig. 3 Number House ›8‹

Sections of the transcribed episodes in Phases and Sub-Phases:

Phase 1		***KLAUS and sönke are finding decompositions for the number houses ›8‹ and ›14‹***
Phase 1.1	[1 – 12]	*KLAUS explains the spelling of numbers.*
Phase 1.2	[13 – 36]	*sönke finds an exchange task to* 4+4.
Phase 1.2.1	[13]	*KLAUS writes the decomposition* 2 + 6.
Phase 1.2.2	[13 – 20]	*sönke writes the decomposition* 5 + 3.
Phase 1.2.3	[21 – 25]	*KLAUS writes the decomposition* 3 + 5.
Phase 1.2.4	[26 – 36]	*sönke proposes the decomposition* 4+4.
Phase 2.1	**[71 – 92]**	***The teacher makes aware of a doubling of the task* 4 + 4 *and of missing decompositions.***
Phase 2.1.1	[71 – 80]	*sönke writes the decomposition* 10 + 4.
Phase 2.1.2	[81 – 92]	*Are both decompositions* 4+4 *and* 4+4 *possible?*
Phase 2.1.2.1	[81 – 83]	*Finding double tasks.*
Phase 2.1.2.2	[84 – 86]	4+4 *and* 4+4 *are not exchange tasks.*
Phase 2.1.2.3	[87 – 92]	*Decompositions with same numbers but in a different order.*

Phase 1: *KLAUS and sönke are finding decompositions for the number houses ›8‹ and ›14‹*

1	*K*	Write down six plus two.
2	s	*(writes down ›d+2‹ on a paper stripe which is supposed to represent one floor of the number house*) That's what I was going to do. You can read thoughts.
3	*K*	Ehm, which letter is that (*points at the ›d‹*)?
4	s	What?
5	*K*	Which number is that?
6	s	<u>Six</u>!
7	*K*	Looks a bit funny.
8	s	Why?
9	*K*	Because a six is written like this (*strokes the table with his forefinger and indicates a 6*). Destroy that (*points at the paper stripe*). That looks like a d.
12	s	(*puts the paper stripe aside and takes a new one, writes down›6 + 2‹ again and hands the pencil to K*)

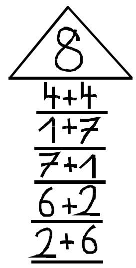

Fig. 4. The first 5 floors of Number House ›8‹

13	*K*	Right, now do, mmm (. . .) It's my turn (*takes a new paper stripe*). May I (*takes the pencil*)? Mmm (*writes down ›2 + 6‹ and gives the pencil to s*). Mmm, what else can we do (*takes a new paper stripe)*?
14	s	Mmm…
15	*K*	Five plus three (*looks at sönke*).
16	s	<u>Yes</u>!
17	*K*	Write down five plus three.
18	s	No.
19	*K*	There is nothing else otherwise.
20	s	(*writes down ›5 + 3‹*) I must be at the end. Then we're done. Then everyone has got the same amount. I am.
21	*K*	Yes? (*takes the pencil*)
22	s	Mmh
23	*K*	Now it's my turn. Now three (. . .) (*writes down ›3 + 5‹ on a new paper stripe*). Is there anything else? Mmm, what else is there? #
24	s	# (*moves each paper stripe up a little*)
25	*K*	Nothing else, is there? (. . .) There is nothing else anymore.
26	s	Do you know what I'm simply going to write down there?
27	*K*	What?
28	s	(*takes a new paper stripe*) I have to once more and then we're done. I'm <u>simply writing down</u> #
29	*K*	# Wait! (. . .) Mmm!
30	s	Four plus four.
31	*K*	Again?
32	s	Yes, we do have every number again. Just the other way round. #
33	*K*	# Yes.
34	s	Four plus four is then also the other way round, just you can't see it. Right, I have a great idea. This four plus four we'll do like this (*writes down ›4 + 4‹*).
35	*K*	<u>Ah</u>, hehe!
36	s	(. . .) Four plus four. Then we have everything.

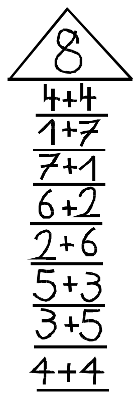

Fig. 5 The first 8 floors of Number House ›8‹

Phase 2.1: The teacher speaks about the doubling of the task 4 + 4 and about incorrect decompositions.

71 *K* We have already finished the one. (*talks to T and points at number house ›8‹*)

72 L Okay. (*has a closer look at the single floors and moves them apart a little bit, he points at two stripes at the same time respectively – he [counts] these*)

73 *K* Ten (. . .)

74 s (*writes down ›10‹*)

75 *K* Plus (. . .)

76 s Plus (. . .) (*writes down a plus sign ›+‹*)

77 *K* Four.

78 s Four. (*writes down a ›4‹*).

79 *K* Now it's my turn. # (*takes the pencil and a new paper stripe*)

80 s # Now.

81 L Ehm, before you continue with the fourteen, I would like you to check here (*points at the number house ›8‹*) again if you perhaps have double tasks. Then you'd have to take these out. #

82 *K* # These two. (*points at the two paper stripes with ›4+4‹ and ›4+4‹*)

83 L Yes.

84 *K* Shall we (. . .) We did them like those (*points at the paper stripes with ›6 + 2‹ and ›2 + 6‹*). Are we not allowed to. Like those, or not?

85 L That means, you thought this was also an exchange task? Yes *(also points at the paper stripes with ›6 + 2‹ and ›2 + 6‹)?and thena at the decomposition downward ›4+4‹)?*

86 *K* Sönke first thought so (*points at S*).

87 L Yes, these (*points at the paper stripe with ›6 + 2‹ and ›2 + 6‹ – looks at K)* are the same numbers, but in a different sequence. #

88 *K* # Yes.

89 L In so far, these are different tasks. But these two (*points at the two paper stripes with ›4 plus 4‹)* you cannot distinguish, can you?

90 *K* Yes, then I'll take this one out (*takes the lower stripe with ›4+4‹ , folds it and puts it aside*).

91 s # But these are different fours (*silently – lets his shoulders fall a little*) #

92 L # Right. Here (*points at number house ›8‹*) are still some missing. Check if you can find one more or two or three. #