Mathematics Knowledge in Teaching about Fraction

Introductory Comments on Liping Ma's
'Knowing and Teaching Elementary Mathematics',

Chapter 3, Generating Representations: Division by Fractions

1. The study reports on the performance of 23 US Teacher and 72 Chinese Teachers in a test to ascertain the teachers' subject matter knowledge and their ability to create or represent meaning in problems on division by fractions. There were two tasks.

How do you solve a problem like this one:

$$1\frac{3}{4} \div \frac{1}{2} = \ldots ?$$

And: *What would you say is a good story or model for* $1\frac{3}{4} \div \frac{1}{2} = \ldots$ *?*

The findings of the study are stark:

Response	US Teachers	Chinese Teachers
COMPUTATION:		
A Correct Algorithm	52%	100%
Incomplete to Wrong Answer	48%	0%
STORY:		
At least one Satisfactory Story	4%	90%
No or Incorrect Story	96%	10%

Would UK teachers have fared better than the US teachers? Would a UK graduate with a Science Batchelor degree, or a Mathematics BSc, have fared better? In this extrapolation, what is the perspective of the mathematics lecturer who teaches on a Mathematics BSc program at a UK university? Do we prepare our mathematics graduate adequately for tasks such as these?

2. Reflecting on the tasks first, and how they relate to each other, it is noted correctly in the article that division of fractions by fractions is among the toughest problems one could expect to meet in elementary mathematics, and for good reasons. When writing $1\frac{3}{4} \div \frac{1}{2}$ it is curious that the symbol to express division in the compound number $1\frac{3}{4}$ is different from the symbol used to express the operation to carry out that division. As if we were to say that $2 \div 5$ results in the number $\frac{2}{5}$. To express the Task 1 in one or the other of the two division meanings we need to write, starting from the middle

$$\frac{\frac{7}{4}}{\frac{1}{2}} = \frac{1\frac{3}{4}}{\frac{1}{2}} = 1\frac{3}{4} \div \frac{1}{2} = (1 + (3 \div 4)) \div (1 \div 2) = (7 \div 4) \div (1 \div 2) = ?$$

Which of the two is easier to handle will depend on personal preference but also on the algorithm used for computation. From the viewpoint of cancellation the left hand side is preferable, while the right could be easier for evaluation by decimals or a computer. The notations are far from universal as already : is used instead of $\div$ elsewhere in Europe while much more complicated symbols are used in Chinese. Lastly note that $\div$ is abolished altogether in professional mathematics, say from university level mathematics onwards, in favour of the – symbol which is more versatile.

Division is difficult as it requires the three inputs dividend, divisor and unit, quite apart from the fact that the first two inputs are fractions themselves. In addition, division is not commutative nor associative. Again one should note that in professional mathematics the operation of division in general tends to be replaced by two simpler operations, namely the multiplication (a prerequisite for division) by the inverse (division in which the dividend is the unit).

Task 1 is achieved by stating an algorithm that computes $1\frac{3}{4} \div \frac{1}{2}$ correctly. It is the easier of the two and this is borne out clearly by the findings. For Task 2 one should view $1\frac{3}{4} \div \frac{1}{2} = ...$ as the solution to a 'story problem' and so finding it is a two-way process: What problem is solved by $1\frac{3}{4} \div \frac{1}{2} = ...$? The various concepts of division stimulate different models so that this two-way relationship has its own interesting dynamics. Another such relationship exists between Tasks 1 and Task 2 as different stories and conceptual models will lead to different algorithmic processes, and *vice versa*.

$$1\frac{3}{4} \div \frac{1}{2}$$

$$\swarrow \qquad\qquad \nwarrow\searrow$$

Computation Algorithm Technique	$\longleftrightarrow$	Problem, Story Model Concept

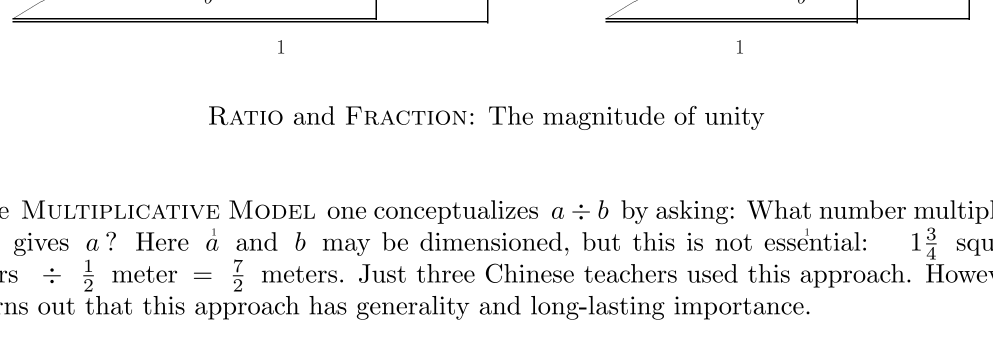
The view that there are 'natural models' for $1\frac{3}{4} \div \frac{1}{2}$ and that finding these would be a a 'natural process' is clearly not tenable, as the data for the American teachers show. Knowing the models is part of subject matter and pedagogical content knowledge, and this part of knowledge is not present in the US participants. And hence it makes little sense to talk about the pedagogical knowledge of the US teachers on the representations of the models. In contrast, for the Chinese teachers there appears to be a canon of concepts for division which lead freely to stories that represent each of the concepts in turn. As is explained in the article, the Chinese teachers' responses correspond with ease to a range of division concepts, and that these occur with varying frequencies rather makes the point, that this canon of concepts is indeed an important part of the teachers' professional formation.

3. So what are these models? In the MEASUREMENT MODEL $a \div b$ is conceptualized by making b the unit of measurement. How often does it fit into a? It is telling that a and b here are dimensioned quantities while the quotient, the number of times, is not : $1\frac{3}{4}$ meters $\div$ $\frac{1}{2}$ meter $=$ $\frac{7}{2}$ times. The measurement model takes us back to addition: how many copies of the unit b add up to a?

In the PARTITIVE MODEL one asks the following: If the b^{th} part of a quantity is a, what is that quantity? Note, here a and the quotient are dimensioned while b is not: $1\frac{3}{4}$ meters $\div\ \frac{1}{2}\ =\ \frac{7}{2}$ meters. Evidently this formulation is problematic, as the 'b^{th} part' already is some kind of inverse so that we could be taking inverses twice, hence leading to a confusion that is well documented in the study: division or multiplication by $\frac{1}{2}$ and multiplication or division by 2 are difficult to keep apart. Here are two sentences to show how this would work in a story: "Half a box of candies weighed $1\frac{3}{4}$ pounds. How much did the whole box weigh?" *(Division by* $\frac{1}{2}$*)* and "Two boxes of candies weighed $1\frac{3}{4}$ pounds. How much did each box weigh?" *(Division by* 2*).*

On balance this model is problematic in its application and does not safeguard well against the confusion between division and multiplication, particularly in the case where fractions are involved. However, it is by far the most favoured model of the Chinese teachers (62 of 80 stories were partitive). One should be interested in the reasons for this. Mathematically the partitive model is *geometric ratio* expressed for numbers. While geometric ratio in itself did not appear explicitly in the study (say, measuring the height of a tree by its shadow in relation to the shadow of a stick of known length) this would explain the popularity of the partitive model. In the Euclidean tradition ratio and proportion play key roles, and this may explain our preference for the partitive model. One should note at this point that ratio is a statement on the relative size of *magnitudes* and for this reason the ratio of ratios (in the terms here: division of fractions by fractions) is in fact not defined until ratio has been turned into magnitude. This requires us to know the magnitude of the unity, see the figure below. Altogether this remains problematic for the application of this fundamental geometric notion.

It would be interesting to understand in which ways the partition model refers to the Chinese mathematical traditions. Euclid arrived in China only in the early 17th century and has had a limited impact.



RATIO and FRACTION: The magnitude of unity

In the MULTIPLICATIVE MODEL one conceptualizes $a \div b$ by asking: What number multiplied by b gives a? Here a and b may be dimensioned, but this is not essential: $1\frac{3}{4}$ square meters $\div\ \frac{1}{2}$ meter $=\ \frac{7}{2}$ meters. Just three Chinese teachers used this approach. However, it turns out that this approach has generality and long-lasting importance.

In this model division is taken back and explained in terms of the nearest previously defined notion, namely multiplication. This process of defining a new notion in terms of existing ones is of first order paradigmatic importance for theoretical mathematics. Of course this is no statement about the pedagogy of that process, and this is why special attention is needed.

To base division by fraction on multiplication just two simple properties are needed: *Any number is equal to a fraction,* $x = \frac{x}{1}$, *for all numbers* x (the Unit Property); and *Two fractions are the same,* $\frac{x}{y} = \frac{a}{b}$, *if and only if* $x \cdot b = y \cdot a$ (Equality). With these two properties in mind the multiplicative models says that "The number x to be found is $x = \frac{x}{1} = \frac{a}{b}$ and so it must be the number for which $x \cdot b = 1 \cdot a = a$."

To show just how useful the two properties are observe how easily the key rule, that *mutiplying dividend and divisor by the same number doesn't change the fraction,* is derived. We have $\frac{a \cdot s}{b \cdot s} = \frac{a}{b}$ because $(a \cdot s) \cdot b = (b \cdot s) \cdot a$ if we allow for the associative and commutative properties. (As we are required to do, the rule does not hold in all generality.)

The article mentions an outstanding example of one teacher who uses the multiplicative model in a rule based fashion to derive that division by fraction is multiplication by the reciprocal:

$$\begin{aligned} 1\tfrac{3}{4} \div \tfrac{1}{2} &= (1\tfrac{3}{4} \times \tfrac{2}{1}) \div (\tfrac{1}{2} \times \tfrac{2}{1}) \\ &= (1\tfrac{3}{4} \times \tfrac{2}{1}) \div 1 \\ &= (1\tfrac{3}{4} \times \tfrac{2}{1}) \\ &= 3\tfrac{1}{2} \end{aligned}$$

This was the only technique used by the US teachers in the computational task. None could give a justification for the validity even if that computation was successful.

4. In the article we were shown various kinds of dysfunctionalities at the vertices and edges of the task triangle. Much has to do with the fact that important parts of subject matter knowledge and related pedagogical content knowledge is not present in the US teacher population. I would suspect that something quite similar could be true in a UK context. While that content, and what I earlier termed 'canon' was once taught and available for students as much as teachers, it no longer is present, and has grown out of the cycle. A teacher now would not recall to have once heard about the three division models if he had not seen them in a degree or training programme since. Hence the spread of this knowledge has been lost.

Rather too little epistemological and pedagogical content is available in UK science degree programmes to reintroduce this kind of knowledge which has importance far beyond the teaching profession. And unfortunately there is little appetite or opportunity to change this. It is reasonable to assume that most UK graduatess can compute $1\frac{3}{4} \div \frac{1}{2}$ correctly but rather more is needed.

Johannes Siemons

School of Mathematics,
University of East Anglia,
Norwich
7 May 2007