Mathematics Knowledge in Teaching about Proof

Comments on Eric J Knuth,

'Teacher's Conceptions of Proof in the Context of Secondary School Mathematics[1]'

1. The paper reports on teachers' conceptions of proof in a small scale study of 17 American high school teachers. This is undertaken in the light of the NCTM (2000) recommendation for extensive experience of proof for all students throughout the entire secondary school curriculum.

As a frame work for proof it is suggested that proof serves the following aims,

- to verify *that* a statement is true,
- to verify *why* a statement is true,
- to communicate mathematical knowledge,
- to discover or create new mathematics,
- to systematize statements into an axiomatic system.

This is much the traditional view on the role of proof that takes its modern roots in the axiomatization project which started in the 1880's. And it is true to say that this program has propelled and underpinned modern mathematics for more than a century.

Contrast this with the 18th and early 19th century view which is encapsulated well in the 1810 edition of the Encyclopaedia Britannica[2], that

> Proof, in *Law* and *Logic,* is that degree of evidence that carries conviction to the mind. It differs from demonstration, which is applicable only to those truths of which the contrary is inconceivable. It differs likewise from probability which produces for the most part nothing more than opinion, while proof produces belief.

Conceptually these two positions are far apart from each other and the field in between is worth exploring. The results of the paper are an interesting part of such an exploration.

2. In the study the teachers would take one of two positions, and were asked to express their concepts of proof as

- the individual who is knowledgeable about mathematics, and
- the individual who is a teacher of school mathematics.

As knowledgeable individuals the majority of teachers tended to view proof as 'a logical or deductive argument that demonstrates the truth of a premise'. Or, that proof provided a convincing argument that something said to be true was indeed true. This view of proof is therefore centered around the formal notions of axiomatization.

[1] Journal of Mathematics Teacher Education, **5**: 61-88, 2002

[2] Encyclopaedia Britannica, 4th Edition, Vol XVII, Edinburgh 1810

For proof in the school context on the other hand several modes of proof emerged, ranging from formal proof, bordering on the formulaic and ritualistic, to less formal and informal proof. A majority of the participants in the study did not consider formal proof as essential, and instead label it as a matter for ‘people in ivory towers’, ‘a topic of study separate from mathematics’, or say that formal proof served only students on advanced classes. Proof in the less formal or informal mode however was thought to be of central importance and needed to be integrated into every class.

3. It is useful to inspect the modes of proof suggested in the paper. While such distinctions are widely made I would like to argue that the axiomatic position is not always helpful and that often it can not be based within the school curriculum.

For the purpose of this note I want to comment on a problem raised in the paper, on the demonstration that the angle sum in the triangle is 180°. First consider tearing off the corners of the triangle and reassembling them into a straight line, as shown in Figure 1. It can be argued that the ‘degree of evidence that carries conviction to the mind‘ was insufficient because the conclusive argument for the straightness of the line is not provided. Other key elements of evidence however are present: a correct understanding of angle which is a difficult concept, different from length or area, and the idea of transforming the problem into one that is closer to a solution. It is precisely this which makes other proofs work and leads to the question why the line indeed is straight. So while this ’proof’ is not carried to conclusion it nevertheless has many essential and valuable ingredients.

Figure 1: Angle sum in the Triangle

A second proof in the paper is given the status of a ‘valid proof’ and is based on the alternating angles created by a line falling on two parallel lines. In the Euclidean tradition this proof is widely accepted as a deduction from the 5 postulates and the

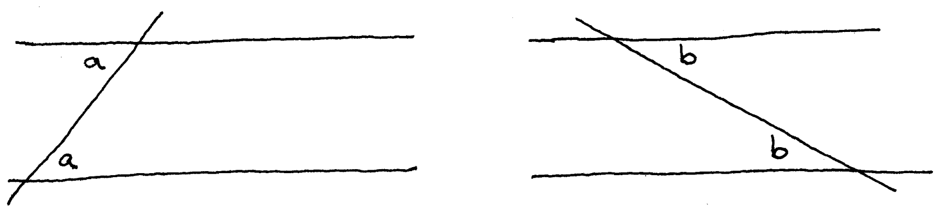

Figure 2: Alternating Angles

common notions in Book 1 of the Elements. In this sense the proof brings conviction and belief, or rather acceptance.

In reality there are significant and well-known difficulties in the Euclidean axiomatics. For instance, the rational plane $\mathbb{Q}^2$ - the points with rational coordinates - is a model for the postulates and the common notions in Book 1. This means that these properties hold true in $\mathbb{Q}^2$. However, already Proposition 1 in Book 1 fails to be true: there is no equilateral triangle in $\mathbb{Q}^2$ over the base vertices $(0,0)$ and $(1,0)$. This says that more assumptions need to be made before Proposition 1 can be proved. In particular, the theorem about the alternating angles can not be deduced from the 5 postulates and the same applies to the theorem on the angle sum, it can not be deduced from the axioms in Book 1. In this sense the 'valid proof' is far from being valid.

These problems were recognized early on and in his Foundations of Geometry (1903) Hilbert has brought geometry back into the fold of axiomatization. A very readable account of Hilbert's work and the full axiomatics of plane geometry can be found in Greenberg's book[3]. Even for the basic geometry of triangles many more axioms are needed. For instance, we require a notion of continuity by which lines intersect in a common point when they cross each other. (This is the reason why Proposition 1 fails in $\mathbb{Q}^2$.) Similarly we are required to say what we mean that two things are they same when they are in different locations. This makes it necessary to include the notion of transformations that keep certain properties invariant, and so on.

One should therefore emphasize that concepts such as continuity and transformation are essential even at the level of elementary geometry. Only when these modern additions have become part of the axiomatic apparatus does it become possible to produce deductive proof in the strict sense. But whether this is appropriate at the level of the secondary school curriculum is a different matter. The sophistication that is required seems well beyond what could be included reasonably. And much of the immediacy which makes Euclid so appealing will be lost in the process.

One lesson to be taken is that proof does not produce truth: It only reveals the extent of the assumptions we make. The dogmatic high ground on proof is difficult to defend and there may be little point in doing so. Proof in the school curriculum needs to find the level of conviction that is appropriate within the respective communities, and it needs to be open about this fact.

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[3]Euclidean and Non-Euclidean Geometries: Development and History, by Marvin Jay Greenberg, Harcourt Publishers, 1993