Developing and deepening mathematical knowledge in teaching: being and knowing

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Mathematics in teaching

In this paper I try to think about mathematical knowledge in teaching as a way of being and acting, avoiding categorisation and acquisition metaphors of knowledge. I think of MKiT as participation in mathematical practices in the classroom, and also during preparation for teaching. Thus development and deepening of knowledge take place through doing mathematics and being mathematical in social contexts in which mathematical habits of mind are embedded, recognised and valued. I shall explain how some of the tasks of teaching can be seen as particular contextual applications of mathematical modes of enquiry.

However, I am not arguing that an enquiry stance about teaching is enough on its own; it is *mathematical* enquiry that I am interested in, and that includes learning the traditional mathematical repertoire. Professional development opportunities that offer only collaborative enquiry as a panacea can be as irrelevant as those that offer only mathematical procedures. A colleague in Alberta has reported that collaborative PD in her school has led to acceptance of a 'lowest common denominator' of practice; a PD session in South Africa offered 16 different rules to factorise quadratics according to relationships between the coefficients. These are extreme but not unusual cases. We can see immediately the nonsense of presenting 'sixteen rules'; they do not relate to the essential activity of seeing quadratics as multiples of binomials, or of seeing the process as revealing roots; nor do they encourage adaptation of general methods in specific cases. Any process of identifying types can go too far and losing overarching insight. This is why I am, in this paper, arguing that a typographical approach to MKiT (knowledge of curriculum, knowledge of students, knowledge of textbooks, etc.) can mask the essential activity within which those nouns connect and inform each other.

Experience of doing mathematics, on one's own and with others, in an environment that encourages listening, questioning and pedagogic reflection (which may be the teacher's own classroom), develops and deepens mathematical knowledge both in and for teaching. One problem with identifying types of knowledge is that we end up with definitions which can be unhelpful for teacher educators – being too unwieldy to fit into institutional constraints– and unhelpful for novices who then get a fragmented sense of what is relevant without yet having the practical perspective to make sense of it.

Teaching as a contextual application of mathematical modes of enquiry

Leikin has written on how the act of teaching, as a listening and interactive teacher, can lead to deeper mathematical knowledge (Leikin 2006; Leikin and Zazkis 2007). Teachers can learn new solutions, new methods, new properties, new distinctions and new questions by listening to learners. In her research, teachers sometimes express these as *not* new, they knew the mathematics already, but they came to *understand* it differently through teaching and they do not necessarily recognise this as 'learning'. However, Leikin and Zazkis present a model which suggests that what is happening is a progressive refinement and formalisation of intuitive knowledge of mathematics, and of how learners engage with the subject. Thus they learn mathematics while teaching, and the learning is embedded in the practices of teaching, and is hence 'mathematics in teaching'.

I now illustrate the importance of mathematical activity by discussing the approach to so-called misconceptions. Knowledge of typical 'misconceptions' is seen as a component in typographies of MKforT. Teachers are sometimes presented with examples of typical errors and invited to think about their responses, or teaching methods which might avoid such errors.

Alternatively, educators can observe how they read the mathematical text of others, a component activity of professional mathematicians as well as by those who teach. Here is an example of mathematical text:

(Omitted from pre-seminar version)

We make sense of it by drawing on our knowledge, talking through the reasoning ourselves, reading carefully where it appears to deviate from what is regarded as 'correct', where we make assumptions, where we jump bits which later turn out to be important, misread, pay attention to the wrong bit of a diagram, try examples for ourselves, and so on. When previous understanding is inadequate for full understanding learners make up bits of reasoning to fill the gaps, or give up and accept the final result parrot-fashion. A teacher who works mathematically can identify the difficult bits of maths, the bits where previous understanding ('met-before' as Tall calls it (Lima & Tall, 2006)) is unhelpful, the places where significant new ways to understand have to be worked on with effort, and where representations do not seem to encapsulate meanings. In this way, the erroneous statements of learners are seen as alternative conceptions generated by mathematical thinking: pattern-seeking, generalising, interpreting, applying. My argument is that knowing how to engage critically with mathematics, communicated in various ways, leads to understanding of how 'errors' are made and therefore reduces the need to learn about individual errors. Furthermore, this approach avoids the promulgation of a 'deficiency' model of learners and replaces it with an understanding that epistemological obstacles are an inherent phenomenon in mathematics arising from the need to learn abstraction, notation and interpretation[1].

In our course we invite PGCE students to collect 'errors' they see students making, and we then have a session in which these are compared and analysed. The process of comparison leads them to classify them, and during this process a transformation takes place. Instead of seeing 'errors' as 'accidents waiting to happen' they see them as manifestations of learners' active engagement with mathematics and offer such classifications as 'over-generalisation', 'reading from left to right', 'substituting something familiar for something unfamiliar' and so on. In other words, they understand 'error' as expressing mathematical habits of mind which can be worked on. This is a huge change from what is typically expressed at interview, that the way to deal with 'errors' is through repeated clear explanation, and takes place through a combination of teaching experience and shared mathematical analysis.

A further route to deeper understanding about 'misconceptions' is to engage teachers in a new area of mathematics[2] and to recognise what happens when their reasoning turns out to be 'incorrect'.

Being mathematical about mathematical knowledge

To introduce some shared experience of being mathematical and knowing mathematics I shall give two examples – one from secondary and one from primary.

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[1] The appearance of the notion of epistemological obstacles in this discussion leads me to question whether refinement and adaptation as described by Leikin & Zazkis are sufficient to learn mathematics, and of course they are not. To learn scientific, counter-intuitive, concepts we have to adopt new ways of thinking and seeing, supported by expert others in their use of language, diagram and other tools.

[2] We often use taxi-cab geometry as an arena which requires new conceptualisation, opportunities to challenge intuition, but low-risk in that nothing depends on understanding it.

For me, it is not static knowledge that I bring to these examples, but observation of my own engagement with them and how this can reveal classic obstacles to understanding.

Taken-as-shared

John Mason and Orit Zaslavsky and I have established, through the editorial processes of a special issue of JMTE, that there are many teacher education practices which can be taken-as-shared (Watson and Mason 2007):

i) teachers engage in mathematical thinking through working on mathematics-related tasks; later they use similar tasks in practice (extended, comparative, multi-stage, realistic, or exercise tasks).

ii) they reflect cooperatively on multiple approaches generated whie working on a task; this helps them understand obstacles perceived by others

iii) they analyse task structures and videos of interactive episodes

iv) they design and try out teaching ideas with learners

v) they observe, analyse and compare teaching (their own and others')

vi) they observe and listen to learners in class or on video

vii) they are encouraged to challenge procedure-dominated approaches

viii) they draw on their own experiences as learners

ix) they are given theoretical constructs to inform practice and make alternative choices available.

In what ways can mathematical knowledge develop and deepen through these kinds of activity?

My underlying assumption is that mathematical knowledge includes its own modes of enquiry.

Discussion of one's own mathematics with others scaffolds a shift from seeing tasks as 'delivering' mathematics to analysing what is afforded and constrained by its variables, structures, similarities, assumptions and generalities and modes of presentation. This is a mathematical analysis. It prepares the way to understand what other people see and interpret in the task, what example spaces are being employed, what limitations are being assumed and so on.

Comparing lessons can be carried out on with attention to several aspects: social, cultural, lesson structure, interactive strategies, overt behaviour and so on, but we are finding that a mathematical analysis is more informative about what is available for students to learn. For example, two lessons, co-planned, both including compass and straight-edge constructions and physical demonstration of locus as 'sets of points' were different mathematical experiences because the students' prior experiences, variation within task, and the emphases through teacher talk were different. Pedagogic comparison throws up some kinds of difference, but not crucial mathematical differences in the nature and perspective of the object of learning. (Hiebert et al. 2003; Marton & Tsui, 2004; Watson, 2008).

Learners' verbal responses and written work can be interpreted as mathematical text. What warrants and backings are being used? What similarities have they drawn on? Their activity can be analysed in relation to the sequence of mathematical interactions in the lesson to make sense of their insights, what makes them stuck, and so on (Al-Murani, 2007; Williams 2005). What is this learner seeing, hearing, enacting, and how do these stimuli lead to a particular response? As a teacher educator all I have to do is draw attention to the possibility of this kind of analysis for interns to recognise it as something to be done. They do not need multiple experiences to learn how to do this, or new vocabulary. They have the mathematical *know-how*[3]. It also seems important that such analysis is discussed; induction into habits and discourse of mathematical analysis of sensory data from a pedagogic view point is at the heart of the development of these skills, making them available for use as teachers.

In our international taken-as-shared list of practices, it seems to be assumed that pre-service teachers do not have experience of extended exploratory mathematical tasks. This is not the case in UK, but it is possible that their experience may have been formulaic and not connected to the development of a mathematical repertoire. Extended tasks challenge views about the nature of mathematics, and this has been widely researched so I will not spend time on this kind of task and related enquiry methods here; they certainly develop something, but this form of enquiry is not always easy to relate to teaching. The perspective I take is rather different, that *all* task-types are available for deep analysis of mathematical affordances, and that such analysis can help teachers develop sensitivity to variations presentation, layout, symbol use, language, diagram and hence to variations in perception, recognition, interpretation on the part of learners. In this way, knowledge of representing, knowledge of possible interpretation, and knowledge of using variables as indicators of the dynamics of mathematical relationships, develops.

Furthermore, this approach pathologises mathematical representations rather than learners in a similar way to the identification of epistemological obstacles: it examines mathematical ideas and their representations as shifts of perception which are inherent in the development of conventional mathematics.

In my research I start from what teachers do, so the classic question about a relationship between personal knowledge and teaching is explored through the practices of teachers.

What is it that non-specialist teachers who have good teaching skills do not do?

In the CMTP project we have been analysing videos of Key Stage 3 lessons from three schools. These are schools whose departments work closely together to improve students' learning. There is considerable shared planning and discussion of lessons. I devised a way of mapping the development of mathematical ideas in lessons which is based on the following key features (Watson 2007):

Focus of episode	**Shifts of mathematical activity**
Teacher makes or elicits declarative/nominal/factual/technical statements	*remembering*
Learners copy, imitate, follow instructions	*developing fluency, reporting/recording actions*
Teacher directs learner perception/attention	*public orientation towards concepts, methods, properties, relationships*
Teacher asks for learner response	*personal orientation towards concepts, methods, properties, relationships*
Discuss implications	*analysis, focus on outcomes and relationships*
Integrate and connect mathematical ideas	*synthesis, connection*
Affirm/ act as if we know …	*rigour, objectification, use*

The lesson episodes can be in any order – there is no implied '7-stage lesson' – but it seems that non-specialist teachers (whose first degree is not in a strongly mathematics-related subject) tend to omit the analysis and synthesis stages, and their affirmation is restricted to use, which is often

[3] There are parallels here with primary teachers whose teaching of other subjects is enriching and imaginative but who cannot see how to apply this in mathematics lessons

superficial or contrived. A further omission for some non-specialist teachers is directing attention towards relationships, as a pre-cursor for thinking about mathematical implications. These are teachers are experienced and have a wide range of established generic skills. I deduce, therefore, that a focus on more interactive and exploratory ways of working, which would relate to the third and fourth rows of this table, is not appropriate for their development. Instead I ask: what mathematical knowledge is required for teachers to think about the implications and integration of learners' mathematical activity for the development of mathematical repertoire and ideas?

Since this varies for every topic being taught, an obvious answer would be that these teachers need more personal experience of the mathematical canon, rather than input about teaching methods.

Why would we think these omissions of perspective are resolved by: studying more maths, or studying special maths?

What do we know about the effects of studying more mathematics on teaching? Mainly what we know is negative: that having higher qualifications does not in itself lead to better teaching (Prestage,1999); that being taught university maths courses as professional development does not necessarily lead to better teaching and can be counter-productive. On the other hand, Potari et al. (2007) found that, in developing interactive teaching skills, those who could make most of learners' ideas were those with richest personal subject knowledge. Let us take an example presented to a year 9 class:

(Omitted from pre-seminar version)

What knowledge is required to construct such a question? One would need mathematical experience at a more advanced level, both of concepts and of how to combine concepts, than the component parts, plus some understanding of analysing complex mathematical statements to find familiar structures.

In the discussion of 'developing and deepening which took place at the September 2007 seminar in this series, we agreed that teachers need "deep and transformative subject knowledge of some areas of mathematics". There is growing interest in the notion of big ideas, key ideas, that provide coherence in the curriculum and this would be one way forward. For example, Hejny has developed what he calls 'restricted arithmetic', essentially mod 99, and primary teaching students spend a long time reconstructing all operations and algebraic structures within this microworld (Stehlikova 2000). This gives experience of what it is like to learn, what it is like to relearn, how to get stuck and unstuck, how to explore, exemplify, generalise and so on but also gets at the essence of meaning of infinite arithmetic, negative numbers, calculation methods, and algebraic structure. Simon has begun to identify key ideas in primary maths and how these connect to hypothetical trajectories of learning the rest of the curriculum (2006).

We have a research link with University of Auckland and have been watching discussions between teacher educators and teachers about the mathematics they are studying to support their development as teachers. One method of analysis of these video discussions is to count the number of times people refer explicitly to ideas being 'big' 'central' 'major' and so on. The NZ project is to find out what happens when 8 teachers are given the opportunity and support to study mathematics for themselves. The support is from mathematics educators within a mathmatics faculty, and much of the study takes place on university mathematics courses. It has to be said that this is an 'education-aware' faculty.

Some findings: Margaret found that she was restructuring her teaching to engage with 'big ideas' in the classroom because that was what had been useful and interesting for her in her own study; Yoko saw for herself a distinction between how she studied for herself and methods she had used before to pass tests; Linda drew distinctions between what you could do for yourself and why you sometimes need a teacher to explain. In our meta-discussions, we found that it seemed as if studying for themselves had led teachers to give more attention to learners' voices and less to their own in the classroom, but we also found that the 'deepest learners' were the better listeners in class. There were several ways in which the experience of studying maths impacted on their practice:

- using their own new knowledge directly (e.g. history; explaining uses of simpler maths in higher maths and outside applications)
- knowing new connections (e.g. logs, indices, bases; graphs and modelling)
- knowing how to study (e.g. how to read text, exemplification, self-explanation, time, revision methods)
- new insights and approaches based more on structure and less on procedure (e.g. big ideas)

My own approach to 'big ideas' currently is to identify some key shifts of perception required to learn mathematics, such as: discrete to continuous, properties to relationships between properties, inductive to deductive reasoning, operations to functions and so on. Whereas some of these have been articulated before in generic terms, particularly in the Australian psychological tradition (e.g. Biggs and Collis 1982; Halford 2003) these sources confirmed that my mathematical analysis of the content was meaningful, rather than informed or triggered that analysis in the first place. It was Vygotsky's articulation of scientific knowledge as that which can only be experienced in formal educational contexts that allowed me to see that knowledge of how and what to teach, and what to listen for in learners, could only be achieved effectively through mathematics itself.

What questions should we be asking about the relationship between personal MK and teaching?

Much of the literature about MK in or for teaching is based on assumptions which I find non-intuitive and sometimes questionable:

1. that the baseline is separate content and method courses and deficient teachers' textbooks for pre-service teachers
2. that teachers know little maths, and that what they do know is in general instrumental and procedural
3. that teacher educators have specialist knowledge about M and MKiT
4. that the processes of analysis and interaction required to teach mathematics are different from those required to learn
5. that studying more maths on maths courses is unhelpful
6. that procedural knowledge can be 'unpacked'

A deficit model is inappropriate: we need to know how the teachers who were doing interesting things (in projects such as SKIMA) came to do those things without higher maths qualifications. For example, a teacher I shall call Sam uses a random generator of multi-digit numbers to ask 'what could the question be?' His highest maths qualification is GCSE and he came to teach in secondary by being redeployed from the role of maths coordinator in a middle school. When using the generator he constantly pushes students for harder and harder examples, and tunes his questions and pushing according to the characteristics of the number itself. How did he become able to do that? By loving mental arithmetic and working on it for himself, he claims.

Can development of mathematical knowledge happen in the teacher-mentor practice?

A further assumption in some of the literature is that the development of MK happens at the HEI institution. Our assumption has been that mentors play a role, and even develop two-way development of MKiT, but we can help by alerting them about how they can do this. In our Communicating About Mathematics in School project we found that some teacher-mentors were indeed engaging with the mathematics of lessons. We elaborated their interactive strategies in a booklet available on the web and given to all our mentors (Watson and De Geest 2003). Briefly, questions and activities clustered among 7 categories, most of which I would describe as concerning mathematical knowledge, its nature and its methods of enquiry:

- What mathematical ideas, operations, meanings, representations, etc. are being taught?
- How are they taught?
- What are the learners' perceptions likely to be?
- What mathematical assumptions and expectations does the teacher have?
- What is the purpose of the tasks, in terms of learners' perceptions of the mathematical ideas etc.?
- How do the ideas develop coherently in the lesson?
- How much variation is offered and why?

From this analysis we recognised that there are methods of lesson analysis, orientated around mathematical activity, that can enhance teachers' learning in practice.

Using existing mathematical experience

The observation that somehow, without over-definition or explicitness about types of knowledge, many people have developed into powerfully effective mathematics teachers leads us, as Oxford, and in my work with teachers everywhere, to start with the recognition, indeed expectation, that teachers learn through participation in practices, rather than through acquisition. Their experience as learners can be unpacked (since it is multi-dimensional) and reconstructed to relate their learner-knowledge to the actual learning that they did. This includes learning the nature of the subject matter, which was more messy than what was eventually accredited, and also more messy than what is often assumed about the limitations of their experience. For example, when asked how to run a revision session they will typically describe something akin to 'show, tell and copy', but when asked *how they revise themselves* a rich variety of practices which form a nexus around the construction of personal narratives emerge. When asked how they learnt differentiation they may describe exposure to procedures and categories of types, but when asked what they talked to each other about while doing homework together they will describe diagrams, arm-waving, argument, wrong turnings, tricky cases, and saying things out loud to themselves as they wrote symbols.

'Unpacking' tacit knowledge does not make it teachable; it is through unpacking their own past and present activity, and observing others do the same, that they learn.

Summary

Further experience of learning mathematics, rather than learning a pedagogic view of mathematics, is a good way to deepen and develop mathematical knowledge in and for teaching.

I have supported this conclusion by:

- Arguing against the use of typographies which are generated by a pedagogic view
- By offering examples of curriculum difficulties which can be understood by analysing their mathematical components and affordances
- Arguing that understanding the activity of doing mathematics provides all the analytical tools needed to think about learning and teaching, and other people's mathematical activity
- Recognising that inherent difficulties in mathematics can be found by a mathematical analysis which avoids pathologising learners and teachers
- By giving examples of teachers who are non-specialists and showing that what they do not do is dependent on personal knowledge; on the other hand, a teacher with a similar

background revealed that he still was an active mathematician within his own sphere if interest

- By giving examples of teachers who studied more maths deliberately to enhance their teaching, and showing how their new learning impacted on teaching
- By showing how past and current experience as a learner can inform teaching.

I also indicated that there was no need to imagine that learning more mathematics while teaching needs to be confined to engagement in courses, but that mentors in school can create opportunities for this.

I am not claiming to have presented a fully joined-up robust argument that it is engagement in the practices of mathematics that enables good teaching, but I do claim to have offered some directions.

Research questions

- What is it that good non-specialist teachers do not do?
- How do teachers whose mathematics teaching is good get to be that way?
- What are the key ideas in mathematics, that make coherent mathematical activity possible?
- What developments of MK take place in practice and how does this happen?

And some further questions we should be asking:

- What are the ways in which mathematics teacher educators in the UK work with pre-service teachers to develop MKifT?
- What is the mathematical background of mathematics teacher educators and mentors in the UK?
- What is the mathematical knowledge required to be an effective mathematics teacher educator?

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